A Note on a Large Production Economy with Increasing Returns and Indivisible Commodities

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Abstract

The existence of Scarf (1986)'s social equilibrium for a large economy with increasing returns (distributive production sets) and indivisible commodities will be proved. We also generalize the Scarf-Oddou's theorem which characterizes the distributive sets by the non-emptiness of the core to the case containing indivisible commodities.

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1 Introduction

In this note, we are concerned with a production economy in which indivisible consumption commodities are produced from the increasing returns to scale technologies. Scarf (1986) defined a class of production sets called the distributive production sets which includes increasing returns to scale (non-convex) technologies. He proposed an equilibrium concept which he called a social equilibrium and showed that it was contained in the core of this economy. He also showed its existence for an economy with finite numbers of the consumers. Hence the core of the economies with distributive production sets are non-empty. Remarkably, he proved a converse of the existence theorem, namely that if the core of the production economies satisfying all of the assumptions for its existence but the distributiveness always exists, then the production sets of those economies are distributive.

Oddou (1976) extended these results to the economies with a continuum of consumers. According to him, the existence of the social equilibrium in a large distributive production economies is a direct consequence of the existence of the competitive equilibrium proved by Hildenbrand (1974). He also proved that the Scarf's converse theorem held in the large economies.

The purpose of this note is to introduce the indivisible commodities into the Scarf-Oddou's existence theorem and the converse theorem for the production economies with the continuum of consumers of the Oddou's type. Its motivation partly comes from the realities; the indivisible commodities are observed more commonly in the industry with the increasing returns such as the automobiles or the electronic industries than those with the decreasing returns such as the agriculture.

From the theoretical point of view, we emphasize that the social equilibrium with the indivisible commodities naturally fits into the set-up of the large economies, and it is appropriate to study this equilibrium concept within the scheme consisting of the continuum of the consumers and the indivisible commodities which was initiated by Mas-Colell (1977). First of all, in this framework, the indivisible commodities are present only in the consumption sets of the consumers. In other words, since the production vectors always appear in the aggregated terms, they are vectors with the coordinates of the real numbers, not integers. Hence we can apply directly the mathematical results due to Scarf and Oddou concerning the distributive sets. This point on the individual *vs.* aggregated terms dichotomy between the consumption sets and the production sets seems not to have been noticed very much, since the most of the papers on this subject, e.g., Mas-Colell (1977) or Yamazaki (1978) have focused only on the exchange economies.

On the other hand, a technical problem arising from the indivisible commodities is that the (upper hemi-) continuity of the individual demand correspondence would generally be lost (see Mas-Colell (1977) and Yamazaki (1978)). Their idea for overcoming this difficulty is now well known. They invoked the so called "smoothing effect of the aggregation" obtained by assumption that the distribution of the consumers' initial endowments is sufficiently "dispersed". However, as Suzuki (1995) showed, this condition would be disturbed by the distribution of the profits if the (convex) production sector existed in the economy. Happily, in the social equilibrium, the production activities are always conducted with the zero profit level, so that this counter effect does not exist. Therefore the dispersed endowments condition of Mas-Colell and Yamazaki works well in the study of the social equilibrium.

Section 2 presents the model and proves the existence of the social equilibrium. The converse of the existence theorem will be proved in Section 3. Section 4 concludes with some remarks.

2 The Model and Results

The commodity space is the $\ell+m$-dimensional Euclidean space $\mathbb{R}^{\ell+m}$, and the nonnegative orthant is denoted by $\mathbb{R}^{\ell+m}_+$. $\mathbb{R}^{\ell+m}_-$ means $-\mathbb{R}^{\ell+m}_+$. $\boldsymbol{x} = (x^t) \geq \mathbf{0}$ means that $x^t \geq 0$ for all t, we denote by $\boldsymbol{x} > \mathbf{0}$ that $\boldsymbol{x} \geq \mathbf{0}$ and $\boldsymbol{x} \neq \mathbf{0}$, and finally $\boldsymbol{x} = (x^t) \gg \mathbf{0}$ means that $x^t > 0$ for all t. We will often denote $(\boldsymbol{x}, \boldsymbol{w})$ or $(\boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^{\ell+m}$, where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^\ell$ and $\boldsymbol{w}, \boldsymbol{z} \in \mathbb{R}^m$. An ℓ-vector $\boldsymbol{x}$ is a consumption vector which is produced from an m-vector $\boldsymbol{z}$. Scarf (1986) called $\boldsymbol{x} = (x^t)$ a consumer vector (x^t is a consumer commodity for $t = 1 \ldots \ell$) and $\boldsymbol{z} = (z^t)$ a producer vector (similarly, z^s is a producer commodity for $s = 1 \ldots m$). Let $\mathscr{I} = \{1 \ldots \ell + m\}$. We denote by $\mathscr{D}$ the set of indexes of the divisible commodities, $\mathscr{D} = \{t \in \mathscr{I} |\ x^t \in \mathbb{R}\}$. Then the commodities in $\mathscr{I} \backslash \mathscr{D}$ are the indivisible commodities which are by definition consumed in the integer units, or $x^t \in \mathbb{Z}$ for $t \in \mathscr{I} \backslash \mathscr{D}$. Throughout this paper, we assume that for simplicity, $\mathscr{D} = \{1, \ell + 1 \ldots \ell + m\}$. There exists only one divisible commodity in the consumer commodities and call it $x^1 \in \mathbb{R}$, the other $\ell - 1$ consumption commodities are consumed by the integer units, $x^t \in \mathbb{Z}$, $t = 2 \ldots \ell$, and all producer commodities are divisible.

All consumers assumed to have an identical consumption set X defined by

$$X = \mathbb{R}_+ \times \mathbb{N}^{\ell-1} \times \mathbb{R}^m_+$$

and then typical consumption vector is denoted by $(\boldsymbol{x}, \boldsymbol{w}) = (x^1, x^2 \ldots x^\ell, w^1 \ldots w^m) \in X$.

Let $(A, \mathcal{A}, \lambda)$ be a measure space of consumers, namely that A is a set of the "names" of consumers, which we assume to be $A = [0, 1]$, the unit interval of $\mathbb{R}$, $\mathcal{A}$ a Borel σ-field of subsets of $A = [0, 1]$ and λ is the Lebesgue measure on $(A, \mathcal{A})$, hence $\lambda(A) = 1$. Each measurable set $C \subset A$ is called a coalition.

As usual, the preference relation $\succsim \subset X \times X$ of a consumer is a complete and transitive binary relation on X. We assume that

(CC) (The only consumer commodities matter): $\xi \sim \zeta$ if $\xi^t = \zeta^t$ for $t = 1 \ldots \ell$,

where $\xi \sim \zeta$ means that $\xi \succsim \zeta$ and $\zeta \succsim \xi$,

(CT) (Continuity): $\succsim$ is a closed subset relative to $X \times X$,

(MT) (Monotonicity for the divisible commodity): for every $\xi \in X$ and every $\epsilon > 0$, $\xi \prec \xi + \epsilon \boldsymbol{b}_1$,

where $\xi \prec \eta$ means that $\eta \succsim \xi$ and not $\xi \succsim \eta$, and $\boldsymbol{b}_t$ is the vector consisting of all 0 components but 1 at the t-th coordinate,

(OD) (Overriding desirability of the divisible commodity): For every $\xi, \eta \in X$, there exists $\delta > 0$ such that $\xi \prec \eta + \delta \boldsymbol{b}_1$.

We denote the set of all preferences by $\mathcal{P}$. Let ω be a λ-integrable function $\omega : A \to X$,

$$a \mapsto \omega(a) = (\boldsymbol{e}(a), \boldsymbol{f}(a)), \ \int_A \omega(a) d\lambda < +\infty,$$
$$e^1(a), f^s(a) \in \mathbb{R}_+, e^t(a) \in \mathbb{N}, t = 2 \ldots \ell, s = 1 \ldots m.$$

The value $\omega(a)$ at $a \in A$ describes the initial endowment vector of the consumer a. From the point of view of individual consumers, the first components of the consumer commodities and the producer commodities are divisible, and the rests are indivisible.

An exchange economy $\mathcal{E}$ is a Borel measurable map

$$\mathcal{E} : A \to \mathcal{P} \times X, \ \ a \mapsto (\succsim_a, \omega(a)),$$

which assigns a consumer a his/her preference and the initial endowment vector.

For the production side of the economy, we follow entirely Scarf (1986), namely that there exists a production possibility set $Y \subset \mathbb{R}^{\ell+m}$ which is available for all coalitions. The following general assumptions will be imposed throughout the paper.

(CL) (Closedness): Y is a closed subset of $\mathbb{R}^{\ell+m}$,

(IA) (Possibility of inaction): $\mathbf{0} \in Y$,

(FD) (Free disposability): $Y + \mathbb{R}^{\ell+m}_- \subset Y$,

and as already mentioned, we will assume that the producer commodities are only used as inputs and not producible,

(NP) (Non-productivity of producer commodities): if $(\boldsymbol{y}, \boldsymbol{z}) \in Y$, then $\boldsymbol{z} \leq \mathbf{0}$.

A pair $(\mathcal{E}, Y)$ of an exchange economy and a production set Y is called a (coalition) production economy or simply an economy. The economy $(\mathcal{E}, Y)$ is considered to be a special case of a coalition production economy defined by Hildenbrand (1974) in which each coalition $C \subset A$ is assigned their own production set $Y(C)$, hence the production technology is described by a correspondence

$$Y : \mathcal{A} \to \mathbb{R}^{\ell+m}, \ \ C \mapsto Y(C).$$

Note that we take a so simple case that the map $Y(C) = Y$, a constant map.

An allocation ϕ of an economy $(\mathcal{E}, Y)$ is an integrable function on A to X which satisfies that $\int_A \phi(a) d\lambda < +\infty$. An allocation ϕ is called feasible if and only if

$$\int_A \phi(a) d\lambda - \int_A \omega(a) d\lambda \in Y.$$

A coalition $C \subset A$ blocks an allocation ϕ if there exists an allocation ψ such that $\phi(a) \prec_a \psi(a)$ a.e. on C, and $\int_C \psi(a) d\lambda - \int_C \omega(a) d\lambda \in Y$. The next definition is standard.

Definition 1 Let $(\mathcal{E}, Y)$ be an economy. The set of all feasible allocations which are not blocked by any non-null coalition is called the core and denoted by $\mathscr{C}(\mathcal{E}, Y)$.

Remark: Note that in the model of large economies (with continuum of traders), the indivisibility of commodities appears in the level of individual consumers and it is seen in the specification of the consumption set. On the other hand, the vectors belonging to production set Y are measured in the aggregate term, and they are represented as if they are perfectly divisible commodities. If the correspondence $Y : \mathcal{A} \to \mathbb{R}^{\ell+m}$ has a Radon-Nikodym derivative η, we can write $Y(C) = \int_A \eta(a) d\lambda$. In this case, $\eta(a) \subset \mathbb{R}^{\ell+m}$ can be interpreted as a production technology of an individual consumer $A \in A$, hence it is natural to impose the indivisible condition on η, or

$$\eta^t(a) \in \mathbb{N}, \quad t = 2 \ldots \ell.$$

The Liapunov's convexity theorem (Hildenbrand (1974, p.62)) makes the vectors $(\boldsymbol{y}, \boldsymbol{z})$ in $Y(C)$ perfectly divisible, $y^t, z^s \in \mathbb{R}, t = 1 \ldots \ell, s = 1 \ldots m$. This formulation of production technologies, however, is not appropriate for the study of the increasing returns, since in this case $\int_A \eta(a) d\lambda$ is always convex.

In order to study the existence of the core and equilibria of the coalition production economy, Scarf (1986) proposed the concept of distributive production sets.

We define a cone Λ in which only the last m components are restricted to be non-negative, $\Lambda = \{(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^{\ell+m} | \boldsymbol{z} \geq \boldsymbol{0}\}$. Then the assumption of the nonproductivity of the producer commodities is written simply as $Y \subset -\Lambda$.

Definition 2 Let Y be a set of $\mathbb{R}^{\ell+m}$ with $\mathbb{R}_-^{\ell+m} \subset Y \subset -\Lambda$. Y is said to be distributive if for any finite number of points $(\boldsymbol{y}_i, \boldsymbol{z}_i) \in Y$ and $\alpha_i \in \mathbb{R}_+$, it follows that $\sum_i \alpha_i (\boldsymbol{y}_i, \boldsymbol{z}_i) \in Y$ whenever $\boldsymbol{z}_i \geq \sum_i \alpha_i \boldsymbol{z}_i$ for all i.

In other words a nonnegative weighted sum will be in Y if it uses more of the producer commodities than any of the original plans. It is easy to see that for the case of two commodities where the commodity 1 is a consumer commodity and the commodity 2 is a producer commodity, the production possibility set Y defined by

$$Y = \{(y, z) \in \mathbb{R}_+ \times \mathbb{R}_- \mid \; y \leq f(-z)\}$$

is distributive if and only if the producer function f is of non-decreasing returns to scale, or $f(\lambda(-z)) \geq \lambda f(-z)$ for $\lambda \geq 1$. Following Lemma due to Scarf (1986) is useful.

Lemma 1 Let Y be a closed distributive set and let $\xi = (\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^{\ell+m}$ be a point with $\xi \notin Y$. Then there is a nonnegative price vector π such that $\pi\xi > 0$ and $\pi\eta \leq 0$ for all $\eta \in Y \cap (\Lambda + \xi)$. Moreover, if $z^s = 0$ for some $s = \ell + 1 \ldots \ell + m$, then we have $\pi^s = 0$.

Scarf (1986) proposed a (non-cooperative or decentralized) market equilibrium concept when the increasing returns prevail.

Definition 3 A triple consisting of an allocation ξ, a production vector $\eta = (\boldsymbol{y}, \boldsymbol{z}) \in Y$ and a price vector $\pi = (\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}_+^{\ell+m}$ is said to consist of a *social equilibrium* of the economy $(\mathcal{E}, Y)$ if and only if

(E-1) $\pi\xi(a) \leq \pi\omega(a)$ and $\xi(a) \succsim_a \xi'$ whenever $\pi\xi' \leq \pi\omega(a)$ a.e,

(E-2) $\pi\eta = 0$ and $\pi\eta' = \pi(\boldsymbol{y}', \boldsymbol{z}') \leq 0$ for all $\eta' \in Y$ with $\boldsymbol{z}' \geq -\int_A \boldsymbol{f}(a)d\lambda$, where $\omega(a) = (\boldsymbol{e}(a), \boldsymbol{f}(a)) \in \mathbb{R}_+^\ell \times \mathbb{R}_+^m$,

(E-3) $\int_A \xi(a)d\lambda - \int_A \omega(a)d\lambda = \eta$.

The condition (E-1) is standard. However, note that in the condition (E-2), the production vector $\eta \in Y$ is chosen so as to yield the zero profit and the equilibrium price vector π requires that the all production plans which are possible under the total resources (producer commodities) available in the economy should have non-positive profit values. As we will see, the fact that the equilibrium profit value is equal to zero is crucial in the proof of Theorem 1.

The most important property of the social equilibrium is that it is contained in the core. Indeed, let $(\pi, \xi(a), \eta)$ be a social equilibrium. Since $\pi\eta = 0$ by the condition (E-2), it follows from (E-1) and (E-3) that $\pi\xi(a) = \pi\omega(a)$ a.e in A. Suppose that the allocation $(\xi(a), \eta)$ be blocked by a non-null coalition $C \subset A$, so that there exist $\xi'(a)$ for almost all $a \in C$ with $\int_C (\xi'(a) - \omega(a))d\lambda \equiv \eta' \in Y$, and $\xi(a) \prec_a \xi'(a)$ a.e. in C. Then using our assumption on preferences,

$$\int_C \pi\xi'(a)d\lambda > \int_C \pi\xi(a)d\lambda = \int_C \pi\omega(a)d\lambda,$$

so that $\pi\eta' > 0$. But $\eta' \in Y$ and $\eta' \geq -\int_C \omega(a)d\lambda \geq -\int_A \omega(a)d\lambda$, which contradicts the condition (E-2) of Definition 3.

The existence of the social equilibrium hence that of the core can be shown by the next theorem.

Theorem 1 Let $(\mathcal{E}, Y)$ be an economy with $\int_A \omega(a)d\lambda = \int_A (\boldsymbol{e}(a), \boldsymbol{f}(a))d\lambda \gg \mathbf{0}$. Suppose that the following conditions hold,

(PD) (Positive divisible endowment): $\lambda\left(\{a \in A |\ \omega^1(a) > 0\}\right) = 1$,

(DD) (Dispersed divisible commodities): for every $\pi = (\pi^t) \in \mathbb{R}_+^{\ell+m}$ with $\pi^1 > 0$,

$$\lambda\left(\{a \in A |\ \sum_{t \in \mathscr{D}} \pi^t \omega^t(a) = w\}\right) = 0 \text{ for each } w \in \mathbb{R},$$

(BP) (Bounded production): the set $Y \cap \{\eta \in \mathbb{R}^{\ell+m} |\ \eta \geq -\int_A \omega(a)d\lambda\}$ is bounded.

Then there exists a social equilibrium (π, ξ, η) for $(\mathcal{E}, Y)$.

Remark The condition (DD) implies that the distribution of perfectly divisible commodity is "dispersed". It was first introduced by Mas-Colell (1977) and generalized by Yamazaki (1978). Its role is well known. The indivisible commodities bring the discontinuous behavior of the individual demand. However, by virtue of the dispersedness condition (DD), the mass of the discontinuous consumers will be measure 0 at each price, hence the aggregate demand correspondence preserves the (upper hemi-) continuity. For details, see Suzuki (2009, Chapter 3).

Proof of Theorem 1 Let K be a compact subset of $\mathbb{R}^{\ell+m}$ which contains in its interior the set of feasible allocations Z defined by

$$Z = \Big\{ (\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^{\ell+m} \Big| \ \boldsymbol{0} \leq \boldsymbol{x} \leq \int_A \boldsymbol{e}(a) d\lambda + \boldsymbol{y}, \\ (\boldsymbol{y}, \boldsymbol{z}) \in Y, \ -\int_A \omega^{\ell+s}(a) d\lambda \leq z^s \leq 0, \ s = 1 \dots m \Big\},$$

and for every positive integer k, define the (truncated) consumption sets by $X_k = X \cap kK$. If we put $A_k = \{a \in A | \omega(a) \in kK\}$, the sets A_k are measurable by the measurability of the map ω and each A_k is of positive measure. For $\omega(a) \in K$ on the set $\{a \in A | \omega(a) \leq \int_A \omega(a) d\lambda\}$, which is of positive measure. Let $\mathcal{A}_k = \{C \cap A_k | C \in \mathcal{A}\}$, and λ_k be the restriction of λ on $\mathcal{A}_k$. Since $(A, \mathcal{A}, \lambda)$ is atomless, so are the measure spaces $(A_k, \mathcal{A}_k, \lambda_k)$.

Let T be the closed convex cone with the vertex at $\boldsymbol{0}$ generated by the set $Y \cap (\Lambda - \int_A \omega(a) d\lambda)$. We claim that $\mathbb{R}^{\ell+m}_- \subset T$. Indeed, it is sufficient to show that $(-\delta^1 \cdots -\delta^{\ell+m}) \in T$ for all sufficiently small $\delta^i > 0$, since T is a cone. This is certainly true since $\omega = \int_A \omega(a) d\lambda \gg \boldsymbol{0}$, and Y satisfies (FD). Next, we shall show that $T \cap \mathbb{R}^{\ell+m}_+ = \{\boldsymbol{0}\}$. In order to prove this, let $\eta = (\boldsymbol{y}, \boldsymbol{z}) \in T \cap \mathbb{R}^{\ell+m}_+$ and $\eta \neq \boldsymbol{0}$. It follows from the assumption (NP) or $\boldsymbol{z} \leq \boldsymbol{0}$ that $\eta = (\boldsymbol{y}, \boldsymbol{0})$, $\boldsymbol{y} > \boldsymbol{0}$. Consider the point $(\boldsymbol{y}, -\int_A \boldsymbol{f}(a) d\lambda)$. Then we have $(\boldsymbol{y}, -\int_A \boldsymbol{f}(a) d\lambda) \in Y$. For if not, we can apply Lemma 1 and obtain a price vector $\pi = (\boldsymbol{p}, \boldsymbol{q}) \geq \boldsymbol{0}$ with $\boldsymbol{p}\boldsymbol{y} - \boldsymbol{q} \int_A \boldsymbol{f}(a) d\lambda > 0$, hence $\pi\eta > 0$ and $\pi\zeta \leq 0$ for all $\zeta \in Y$ with $\zeta^{\ell+s} \geq -\int_A f^s(a) d\lambda$, $s = 1 \dots m$. But this implies that $\pi\zeta \leq 0$ for all $\zeta \in T$ and therefore $\pi\eta \leq 0$. This contradicts the previous inequality. Hence $(\boldsymbol{y}, -\int_A \boldsymbol{f}(a) d\lambda) \in Y$. Repeating the same argument on $r\eta$ instead of η for all $r > 0$, we see that $(r\boldsymbol{y}, -\int_A \boldsymbol{f}(a) d\lambda) \in Y$ for all $r > 0$, and this contradicts the assumption (BP).

For each $t \in \mathscr{D}$, let $\beta_k = (-1, 1/k \dots 1/k)$, $k = 1, 2 \dots$. Let T_k be the closed convex cone with the vertex at $\boldsymbol{0}$ which is generated by $T \cup \{\beta_k\}$. Since $\beta_k \to (-1, 0 \dots 0) \in \mathbb{R}^{\ell+m}_- \subset T$, it is obvious that $T_1 \supset T_2 \cdots \to T$ in the topology of closed convergence (Hildenbrand (1974, p.18)). For each k, we define the price space Π_k as the polar of the set T_k, or $\Pi_k = \{\pi = (\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^{\ell+m} | \ \pi\zeta \leq 0 \text{ for all } \zeta \in T_k\}$. By the construction, we obviously have $\pi^1 > 0$ for $\pi \in \Pi_k$.

For each $a \in A_k$, we define the (truncated) demand relation

$$\phi_k(a, \pi) = \{\xi \in X_k | \ \pi\xi \leq \pi\omega(a) \ \text{ and } \ \xi' \precsim_a \xi \ \text{ whenever } \ \pi\xi' \leq \pi\omega(a)\},$$

for every $\pi \in \Pi_k$. Since X_k is compact, it is standard to verify that $\phi_k(a, \pi) \neq \emptyset$. Proposition 2 of Hildenbrand (1974, p.102) then applies also to the case of indivisible commodities, hence $\phi_k(a, \pi)$

has a measurable graph. Hence we can define the (truncated) mean demand by

$$\Phi_k(\pi) = \int_{A_k} \phi_k(a,\pi)d\lambda \ \text{ for } \ \pi \in \Pi_k.$$

It is well known hat $\Phi(\pi)$ is nonempty, compact and convex valued (Hildenbrand (1974, p.62)). We will show that it is upper hemi-continuous. Since X_k is compact, it suffices to verify that it is a closed correspondence. Let (π_n, ξ_n) be a sequence in $\Pi_k \times X_k$ such that $(\pi_n, \xi_n) \to (\pi, \xi)$ and $\xi_n \in \Phi_k(\pi_n)$ for all n. We must show that $\xi \in \Phi_k(\pi)$. For each n, there exists an integrable function g_n of A_k to X_k with $g_n(a) \in \phi_k(a, \pi_n)$ a.e. in A_k and $\int_A g_n(a)d\lambda = \xi_n$. For the price vector $\pi = (\boldsymbol{p}, \boldsymbol{q}) \in \Pi_k$, define a subset $A_k(\pi)$ of A_k by

$$A_k(\pi) = \{a \in A_k | \ \boldsymbol{p}(0, \boldsymbol{z}) = \pi\omega(a) \ \text{ for some } \ \boldsymbol{z} \in \mathbb{N}^{\ell-1}\}.$$

Since $A_k(\pi) = \{a \in A_k| \ p^1e^1 + \sum_{s=1}^m q^s f^s(a) = \sum_{t=2}^\ell p^t(z^t - e^t(a))\}$ and z^t, $e^t(a) \in \mathbb{N}$ for $t = 2 \dots \ell$, it follows from the assumption (DD) that $\lambda(A_k(\pi)) = 0$, since $p^1 > 0$. First we shall show that

$$Ls(g_n(a)) \subset \phi_k(a,\pi) \ \text{ for a.e. in } \ A_k \backslash A_k(\pi). \tag{1}$$

Let $a \in A_k \backslash A_k(\pi)$ be such that $g_n(a) \in \phi_k(a, \pi_n)$ for every n and let $g_n(a) \to g(a) \in X_k$. Since $\pi_n g_n(a) \le \pi_n \omega(a)$ for all n, $\pi g(a) \le \pi\omega(a)$. Take $\xi' \in X_k$ with $\pi\xi' \le \pi\omega(a)$. We want to show that $g(a) \succsim_a \xi'$. By the assumption (CC), we can assume without loss of generality that $\xi' = (\boldsymbol{x}, \boldsymbol{0})$, $\boldsymbol{x} \in \mathbb{R}^\ell$, $\boldsymbol{0} \in \mathbb{R}^m$. For $\boldsymbol{x} = (x^1 \dots x^\ell) \in \mathbb{R}^\ell_+$ such that $\boldsymbol{px} < \pi\omega(a)$, we have $\pi_n\xi' < \pi_n\omega(a)$ for all n sufficiently large. By the continuity of $\succsim_a$, one obtains $g(a) \succsim_a \xi'$. If $\boldsymbol{px} = \pi\omega(a)$, there exist $x^1_i (i = 1, 2, \dots)$ such that $x^1_1 < x^1_2 < \cdots \to x^1$, since $a \notin A_k(\pi)$. Let ξ_i be the vector which is equal to ξ' but the first coordinate is replaced by x^1_i. Then it follows from $\xi_i \to \xi'$ that $g(a) \succsim_a \xi'$ by the continuity of $\succsim_a$. The claim (1) then implies that

$$\xi \in \int_{A_k} Ls(g_n(a) d\lambda_k \subset \int_{A_k} \phi_k(a,\pi) d\lambda_k = \Phi_k(\pi).$$

We can then apply the fixed point theorem (Hildenbrand (1974, p.39))

Lemma 2. Let Π be a closed convex cone with the vertex $\boldsymbol{0}$ in $\mathbb{R}^{\ell+m}$ which is not a linear subspace. If the correspondence ζ of Π into $\mathbb{R}^{\ell+m}$ is nonempty, compact and convex valued and upper hemi-continuous, and satisfies $\pi\zeta(\pi) \le 0$ for every $\pi \in \Pi$, then there exists $\pi^* \in \Pi$ with $\pi^* \ne \boldsymbol{0}$ such that $\zeta(\pi^*) \in T$, where we recall that T is the polar of the set Π,

Let $\zeta_k(\pi) = \int_{A_k} \phi_k(a,\pi)d\lambda_k - \int_{A_k} \omega(a)d\lambda_k$. Then for each k, there exists a price vector $\pi_k \in \Pi_k$ and an integrable function $g_k(\cdot)$ of A_k to $\mathbb{R}^{\ell+m}$ such that

$$g_k(a) \in \phi_k(a, \pi_k) \ \text{ a.e. in} A_k \tag{2}$$

$$\eta_k \equiv \int_{A_k} g_k(a)d\lambda_k - \int_{A_k} \omega(a)d\lambda_k \in T_k, \ \ k = 1, 2 \dots. \tag{3}$$

We extend the domain A_k of f_k to A by defining $g_k(a) = \omega(a)$ for $a \in A \backslash A_k$. Then the condition (14) is replaced by

$$\int_I g_k(a)d\lambda - \int_I \omega(a)d\lambda \in T_k, \quad k = 1, 2 \ldots. \tag{4}$$

We now obtain a sequence $(\xi_k = \int_A g_k(a)d\lambda, \eta_k, \pi_k) \in X \times T_k \times \Pi_k \subset X \times T_k \times \Pi$. Without loss of generality, we can assume that $\pi_k \to \pi \in \Pi$. We claim that $\{\eta_k\}$ is bounded. Suppose not. Then $\|\eta_k\| \to +\infty$. Let $\hat{\eta}_k = \eta_k / \|\eta_k\|$. Since $\|\hat{\eta_k}\| = 1$ for all k, we can assume without loss of generality that $\hat{\eta}_k \to \hat{\eta}$. Since T_k is a cone, $\hat{\eta}_k \in T_k$ for all k. $T_k \to T$ in the topology of closed convergence implies that $\hat{\eta} \in T$. Since X is bounded from below, $\hat{\eta} \geq \mathbf{0}$. This contradicts $T \cap \mathbb{R}^{\ell+m}_+ = \{\mathbf{0}\}$. We conclude that $\eta_k \to \eta' \in T$, and consequently the sequence $\{\xi_k\} = \{\int_A g_k(a)d\lambda\} = \{\int_A \omega(a)d\lambda + \eta_k\}$ is bounded. Then by the Fatou's lemma in ℓ-dimensions (Hildenbrand (1974, p.69)), there exists an integrable function g of A to $\mathbb{R}^{\ell+m}$ such that

$$g(a) \in Ls(g_k(a)) \text{ a.e. in } A, \tag{5}$$

$$\int_A g(a)d\lambda \leq \int_A \omega(a)d\lambda + \eta'. \tag{6}$$

Now for almost all $a \in A$, there exists a positive integer $k(a)$ such that

$$k > k(a) \text{ implies } g_k(a) \in \phi_k(a, \pi_k), \tag{7}$$

for we can take $k(a)$ as a positive integer not smaller than $\|\omega(a)\|/min\{\int_A \omega^t(a)d\lambda + y^t|\ t = 1 \ldots \ell\}$. Such an integer exists, since $\int_A \omega(a)d\lambda \gg \mathbf{0}$ and $\boldsymbol{y} \geq \mathbf{0}$. Then $0 \leq \omega^t(a) \leq \|\omega(a)\| \leq k(a)min\{\int_A \omega^t(a)d\lambda + y^t|\ t = 1 \ldots \ell\} \leq k(a)(\int_A \omega^t(a)d\lambda + y^t)$, $t = 1 \ldots \ell$.

We next show that

$$g(a) \in \phi(a, \pi) \text{ a.e in } A. \tag{8}$$

First we shall show that $\pi^1 > 0$. By the assumption (PD), this implies that $\pi\omega(a) > 0$ a.e. in A. Suppose on the contrary that $\pi^1 = 0$. Since $\pi > 0$ and $\int_A \omega(a)d\lambda \gg \mathbf{0}$, we have $\pi \int_A \omega(a)d\lambda > 0$. Hence $\pi\omega(a) > 0$ on a set $B \subset A$ of positive measure. For some $t \neq 1$, $\pi^t\omega^t(a) > 0$ on B. Take $\epsilon > 0$ such that $\omega^t(a) - \epsilon \geq 0$, and for each k, take $\delta_k > 0$ such that $\pi^1_k\delta_k - \pi^t_k\epsilon = 0$. Define $\boldsymbol{z}_k = \omega(a) + \delta_k\boldsymbol{b}_1 - \epsilon\boldsymbol{b}_t$, where $\boldsymbol{b}_t = (0 \ldots 0, 1, 0 \ldots 0)$ and 1 is at the t'th position. Then one obtains $\pi_k\boldsymbol{z}_k = \pi_k\omega(a)$ for all k and $\delta_k = \pi^t_k\epsilon/\pi^1_k \to +\infty$, hence $g_k(a) \prec_a \boldsymbol{z}_k$ for k large enough by the assumption (OD). This contradicts $g_k(a) \in \phi_k(a, \pi_k)$.

It is obvious from (7) that $\pi g(a) \leq \pi\omega(a)$. Let $a \in A \backslash A(\pi)$, where $A(\pi) = \{a \in A|\ \boldsymbol{p}(0, \boldsymbol{z}) = \pi\omega(a)$ for some $\boldsymbol{z} \in \mathbb{N}^{\ell-1}\}$. For $\xi \in X$ with $\pi\xi < \pi\omega(a)$, we have $g(a) \succsim_a \xi$ as before. If $\pi\xi = \pi\omega(a)$, there exists a sequence ξ_i with $\xi_i \to \xi$ from below and $\pi\xi_i < \pi\omega(a)$, since $a \notin A(\pi)$. Then $g(a) \succsim_a \xi_i$, hence $g(a) \succsim_a \xi$ by the continuity of $\succsim_a$. Since $\lambda(A(\pi)) = 0$, the claim (8) is verified.

Let $\zeta = \int_A g(a)d\lambda - \int_A \omega(a)d\lambda - \eta'$ and $\eta = \eta' + \zeta$. Then $\eta \in T$, since $\mathbb{R}_-^{\ell+m} \subset T$ and it follows that

$$\int_A g(a) = \int_A \omega(a) + \eta. \tag{9}$$

Since the preferences are locally non-satiated by the assumption (MT), we have $\pi g(a) = \pi\omega(a)$ a.e. in A. Hence $\pi\eta = 0$. Since $\pi \in \Pi$, we see that $\pi\zeta \leq 0$ for all $\zeta \in T$. It follows from the last condition that $\pi\zeta \leq 0$ for all $\zeta = (\zeta^t) \in Y$ with $\zeta^{\ell+s} \geq -\int_A \omega^{\ell+s}$, $s = 1\ldots m$.

Finally we complete the proof by showing

$$\eta = (\boldsymbol{y}, \boldsymbol{z}) \in Y. \tag{10}$$

It is obvious that $(\boldsymbol{y}, \boldsymbol{z}) = \eta \geq -\int_A \omega(a)d\lambda = (-\int_A \boldsymbol{e}(a)d\lambda, -\int_A \boldsymbol{f}(a)d\lambda) \in \mathbb{R}^\ell \times \mathbb{R}^m$, since $\int_A g(a)d\lambda \geq \mathbf{0}$. We now claim that if $z^s > -\int_A f^s(a)d\lambda$, then $\pi^{\ell+s} = 0$, $s = 1\ldots m$. To see this, suppose that $z^1 > -\int_A f^1(a)d\lambda$ and $\pi^{\ell+1} = q^1 > 0$. Then $\int_A g^{\ell+1}(a)d\lambda > 0$ so that $g^{\ell+1}(a) > 0$ on a set with positive measure. On this set, let $\epsilon(a) = \pi^{\ell+1}g^{\ell+1}/\pi^1$ and define $h(a) = (g^1(a)+\epsilon(a), g^2(a)\ldots g^\ell(a), 0\ldots 0)$. Then it follows that $\pi h(a) = \pi\omega(a)$ and $g(a) \prec_a h(a)$ by the assumption (MT), a contradiction. This means that if $z^s > -\int_A f^s(a)d\lambda$ for $s = 1\ldots m$, then we can reduce z^s without disturbing the preferences, maintaining $\pi\eta = 0$, and the modified η will be still be in T, since $\mathbb{R}_-^{\ell+m} \subset T$. Let us assume that this has been done and that $\boldsymbol{z} = -\int_A \boldsymbol{f}(a)d\lambda$.

Now suppose that $\eta \notin Y$. Using Lemma 1, we obtain a non-zero vector ρ with $\rho\eta > 0$ and $\rho\zeta \leq 0$ for all $\zeta \in T$ with $\zeta^{\ell+s} \geq -\int_A \omega^{\ell+s}(a)d\lambda$, $s = 1\ldots m$. But this implies that $\rho\zeta \leq 0$ for all $\zeta \in T$. Since $\eta \in T$, this is a contradiction and the proof is complete. ∎

Remark As we have already pointed out, the equilibrium production $\eta \in Y$ satisfies that $\pi\eta = 0$. Hence the definition of $A_k(\pi)$ (and $A(\pi)$) is independent of the production vector, that is to say, the income distribution of the consumers is "dispersed" or $\lambda(A_k(\pi)) = 0$ whenever the endowment distribution is dispersed, no matter what the production level at the equilibrium is. If the income distribution contained the profit from the production activity, it could happen that dispersed endowment distribution is disturbed and the measure of the set $A_k(\pi)$ could not be zero anymore. This situation appears in the competitive equilibrium with convex production sets (Suzuki (1995) and (2009, Chapter 3)).

3 A Converse of the Existence Theorem

In this section, we discuss a converse of Theorem 1 which was first proved by Scarf (1986) in an economy with a finite number of consumers and was generalized by Oddou (1976) to an economy with a measure space of the consumers. The end of this section is Theorem 2, in which we will introduce the indivisible commodities to the Scarf-Oddou's theorem.

First we give a definition of the effective production plan.

Definition 4. Let Y be a production set satisfying the assumption (NP). A production plan $\eta_* = (\boldsymbol{y}_*, \boldsymbol{z}_*)$ is efficient if there exists no $\eta = (\boldsymbol{y}, \boldsymbol{z}) \in Y$ with $\eta_* \leq \eta$ and $\boldsymbol{z}_* \leq \boldsymbol{z} \ (\leq \mathbf{0})$.

In words, a production plan is efficient if we can not produce more consumer goods from less producer commodities. The next lemma which characterizes the distributive sets, also due to Scarf (1986) is useful.

Lemma 3. Let $Y \subset \mathbb{R}^{\ell+m}$ be a production set and $\eta_* = (\boldsymbol{y}_*, \boldsymbol{z}_*)$ be an efficient production plan. Suppose that there exists a price vector $\pi \in \mathbb{R}^{\ell+m}_+$ such that $\pi\eta_* = 0$ and $\pi\eta \leq 0$ for all $\eta = (\boldsymbol{y}, \boldsymbol{z}) \in Y$ such that $\boldsymbol{z}_* \leq \boldsymbol{z} \ (\leq \mathbf{0})$. Then Y is a distributive set.

In order to prove Theorem 2, we add one more assumption on the production set.

(CS) (Convex section): For every $\boldsymbol{z}_0 \in \mathbb{R}^m_-$, the set $\{\boldsymbol{y} \in \mathbb{R}^\ell | \ (\boldsymbol{y}, \boldsymbol{z}_0) \in Y\}$ is convex subset of $\mathbb{R}^\ell$.

Recall that $\mathscr{I} = \{1 \dots m\}$ and $\mathscr{D}$ be the set of indexes of the divisible commodities, $\mathscr{D} = \{t \in \mathscr{D} | \ x^t \in \mathbb{R}\}$. We now state our main result of this section, which is a converse of Theorem 1.

Theorem 2. Let Y be a closed subset of $\mathbb{R}^{\ell+m}$ containing $\mathbf{0}$ and having convex sections (CS), in which free disposal (FD) and non-productivity of producer commodities (NP) are satisfied. Suppose that for every exchange economy $\mathcal{E} : A \to X$ with the assumptions of (PD), (BP) and the dispersed divisible endowments,

(DD) for every $\pi = (\pi^t) \in \mathbb{R}^{\ell+m}_+$ with $\pi^1 > 0$,

$$\lambda\left(\{a \in A| \ \sum_{t \in \mathscr{D}} \pi^t \omega^t(a) = w\}\right) = 0 \ \text{ for each } \ w \in \mathbb{R},$$

the production economy $(\mathcal{E}, Y)$ has a nonempty core, $\mathscr{C}(\mathcal{E}, Y) \neq \emptyset$. Then the production set Y is distributive.

Proof. The proof will proceed along the similar line as that of Oddou (1976). In order to apply Lemma 3, let $\eta_* = (\boldsymbol{y}_*, \boldsymbol{z}_*)$ be an efficient production vector. First we assume that $\boldsymbol{z}_* \ll \mathbf{0}$, and find a price vector $\pi = (\boldsymbol{p}, \boldsymbol{q})$ such that $\pi\eta_* = 0$ and $\pi\eta \leq 0$ for all $\eta = (\boldsymbol{y}, \boldsymbol{z})$ with $\boldsymbol{z}_* \leq \boldsymbol{z} (\leq \mathbf{0})$. Since the set $Y(\boldsymbol{z}_*) = \{\boldsymbol{y} \in \mathbb{R}^\ell | \ (\boldsymbol{y}, \boldsymbol{z}_*) \in Y\}$ is convex, there exists a price vector $\boldsymbol{p} \in \mathbb{R}^\ell \backslash \{\mathbf{0}\}$ with $\boldsymbol{p}\boldsymbol{y}_* \geq \boldsymbol{p}\boldsymbol{y}$ for all $\boldsymbol{y} \in Y(\boldsymbol{z}_*)$. Since $\boldsymbol{y} + \mathbb{R}^\ell_- \subset Y(\boldsymbol{z}_*)$, we have $\boldsymbol{p} \geq \mathbf{0}$. Since $\boldsymbol{p} \neq \mathbf{0}$, we can assume without loss of generality that $p^1 > 0$.

Let $\epsilon > 0$ and define

$$E = \{\boldsymbol{e} = (e^t) \in \mathbb{R} \times \mathbb{N}^{\ell-1} \mid \ e^1 \geq \epsilon, e^t \geq 1, \ \ t = 2 \dots \ell\}.$$

In the following, we will use an integrable function $\theta : A \to \mathbb{R}_+$ such that $\int_A \theta(a) d\lambda = 1$ and $\inf\{\theta(a) \mid \ a \in A\} \equiv \theta_* > 0$, and $\theta(a) \neq \theta(a')$ for $a \neq a'$.

Take $\boldsymbol{e} = (e^1 \dots e^\ell) \in E$. We then define $\boldsymbol{e}_* = (\theta_* e^1, e^2 \dots e^\ell)$, $B = m^{-1} max\{-z_*^s \mid \ s = 1 \dots m\}^{-1}$, and

$$\begin{aligned} Q_{\boldsymbol{e}} = \{\boldsymbol{q} \in \mathbb{R}_+^m \mid \ & \boldsymbol{q}\boldsymbol{z}_* = \boldsymbol{p}\boldsymbol{y}_*, \boldsymbol{q}\boldsymbol{z} \geq \boldsymbol{p}\boldsymbol{y} \ \text{ for all } \ (\boldsymbol{y}, \boldsymbol{z}) \in Y \\ & \text{with } \ \boldsymbol{z}_* \leq \boldsymbol{z} \ \ (\leq \boldsymbol{0}) \ \text{ and } \ \boldsymbol{y} \geq -B\|\boldsymbol{z}\|\boldsymbol{e}_*\}. \end{aligned}$$

Since $\boldsymbol{z}_* \ll \boldsymbol{0}$, $Q_{\boldsymbol{e}}$ is bounded. Obviously, it is a closed subset of $\mathbb{R}^m$, hence it is compact. We will show that for each $\boldsymbol{e} \in E$, $Q_{\boldsymbol{e}} \neq \emptyset$.

Let$\{A_1 \dots A_m\}$ be a measurable partition of $A = [0, 1]$ defined by

$$A_s = \left[\frac{s-1}{m}, \frac{s}{m}\right), s = 1 \dots m-1, \ \ A_m = \left[\frac{m-1}{m}, 1\right].$$

We then have $\lambda(A_s) = 1/m$, $s = 1 \dots m$, and for each s, we define an integrable function $f^s : A \to \mathbb{R}_+$ by

$$f^s(a) = -m z_*^s \chi_{A_s}(a), \ \ s = 1 \dots m,$$

where $\chi_C(a) = 1$ for $a \in C$ and $\chi_C(a) = 0$ otherwise, for every $C \in \mathcal{A}$. Setting $\boldsymbol{f}(a) = (f^1(a) \dots f^m(a))$, it follows that $\int_A \boldsymbol{f}(a) d\lambda = -\boldsymbol{z}_*$.

We construct an exchange economy as follows. For each $a \in A$, we define the consumption set of a by

$$X = \mathbb{R}_+ \times \mathbb{N}^{\ell-1} \times \mathbb{R}_+^m,$$

the utility function $u_a : X \to \mathbb{R}$ by

$$u_a(\boldsymbol{x}, \boldsymbol{w}) = \boldsymbol{p}\boldsymbol{x},$$

and finally, the initial endowment $\omega(a) = (\boldsymbol{e}(a), \boldsymbol{f}(a))$ by

$$\boldsymbol{e}(a) = (\theta(a) e^1, e^2 \dots e^\ell),$$

where $\boldsymbol{e} = (e^1 \dots e^\ell) \in E$. We then have gotten an exchange economy $\mathcal{E}(a) = (u_a, \omega(a))$. Note that for the production economy $(\mathcal{E}, Y)$, $\mathscr{D} = \{1, \ell+1 \dots \ell+m\}$ and the endowment distribution of the divisible commodities is dispersed. Moreover, since $p^1 > 0$, the preference $u_a(\boldsymbol{x}, \boldsymbol{w}) = \boldsymbol{p}\boldsymbol{x}$ satisfies the monotonicity and the overriding desirability. Then by the assumption of Theorem 2, the core of the economy $(\mathcal{E}, Y)$ is nonempty, $\mathscr{C}(\mathcal{E}, Y) \neq \emptyset$.

Therefore there exists an allocation $\xi : A \to X$ with $\xi(a) = (\boldsymbol{x}(a), \boldsymbol{0})$ satisfying

$$\int_A (\xi(a) - \omega(a)) d\lambda \in Y$$

and for every $C \in \mathcal{A}$, $\boldsymbol{p}\boldsymbol{y} \le \boldsymbol{p}\int_C (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda$ whenever $(\boldsymbol{y}, -\int_C \boldsymbol{f}(a)d\lambda) \in Y$ and $\boldsymbol{y} + \int_C \boldsymbol{e}(a)d\lambda \in \mathbb{R}^\ell_+$. It follows from this with setting $\boldsymbol{y} = \boldsymbol{0}$ that $\boldsymbol{p}\int_C (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda \ge 0$ for every $C \in \mathcal{A}$.

Take $\boldsymbol{z} = (z^t) \in \mathbb{R}^m_-$ with $\boldsymbol{z}_* \le \boldsymbol{z} < \boldsymbol{0}$, and denote $\zeta^s = z^s / z^s_*$ and $-q^s z^s_* = \boldsymbol{p}\int_{A_s} (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda$. We claim that for each $s = 1 \dots m$, there exists a measurable set $C_s \subset A_s$ such that $\lambda(C_s) = \zeta^s \lambda(A_s)$ and $\int_{C_s} (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda \le \zeta^s \int_{A_s} (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda$.

Indeed, for each positive integer n, let $\{A^1_s \dots A^{2^n}_s\}$ be the division of A_s by 2^n closed subsets with $A_s = \cup_{i=1}^{2^n} A^i_s$ and $\lambda(A^i_s) = 1/m2^n$, $i = 1 \dots 2^n$. Let k_s be the smallest integer with $2^n \zeta^s \le k_s$. Denoting $I^i_s = \int_{A^i_s} (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda$, we can assume without loss of generality that $I^1_s \le I^2_s \le \cdots \le I^{2^n}_s$. Then for every n, it follows that

$$\lambda\left(\sum_{i=1}^{k_s} A^i_s\right) = k_s/m2^n,$$

and

$$\sum_{i=1}^{k_s} \int_{A^i_s} (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda \le (k_s/2^n) \int_{A_s} (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda.$$

Since the set $\mathscr{F}_0(A)$ of nonempty closed subsets of A with the topology induced from the Hausdorff metric is compact, we have a closed set C_s with $\cup_{i=1}^{k_s} A^i_s \to C_s$ as $n \to \infty$. We then have

$$\lambda(C_s) = \zeta^s \lambda(A_s),$$

and

$$\int_{C_s} (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda \le \zeta^s \int_{A_s} (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda.$$

Let $C = \cup_{s=1}^m C_s$. It follows that

$$\int_C \boldsymbol{f}(a)d\lambda = \sum_{s=1}^m \zeta^s \int_{A_s} \boldsymbol{f}(a)d\lambda = \sum_{s=1}^m \zeta^s(-z^s_*)\boldsymbol{b}_s = -\boldsymbol{z},$$

where $\boldsymbol{b}_s = (0 \dots 0, 1, 0 \dots 0)$ and 1 is at the s'th position. On the other hand,

$$\boldsymbol{p}\int_C (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda \le \sum_{i=s}^m \zeta^s \boldsymbol{p} \int_{A_s} (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda = -\sum_{s=1}^m \zeta^s q^s z^s_* = -\sum_{s=1}^m q^s z^s = -\boldsymbol{q}\boldsymbol{z},$$

and

$$\int_C \boldsymbol{e}(a)d\lambda \ge \lambda(C)\boldsymbol{e}_* = \sum_{s=1}^m \zeta^s \lambda(A_s)\boldsymbol{e}_* = (1/m)\sum_{s=1}^m \frac{-z^s}{-z^s_*}\boldsymbol{e}_* \ge B\|\boldsymbol{z}\|\boldsymbol{e}_*.$$

Therefore $(\boldsymbol{y}, \boldsymbol{z}) \in Y$ and $\boldsymbol{y} + B\|\boldsymbol{z}\|\boldsymbol{e}_* \ge \boldsymbol{0}$ imply that

$$\boldsymbol{p}\boldsymbol{y} \le \boldsymbol{p}\int_C (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda \le -\boldsymbol{q}\boldsymbol{z}.$$

We want to show that $\boldsymbol{p}\boldsymbol{y}_* + \boldsymbol{q}\boldsymbol{z}_* = 0$. It follows from $\int_A (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda \in Y(\boldsymbol{z}_*)$ that

$$\boldsymbol{p}\boldsymbol{y}_* \geq \boldsymbol{p} \int_A (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda = -\boldsymbol{q}\boldsymbol{z}_*.$$

If $\boldsymbol{y}_* + \int_A \boldsymbol{e}(a)d\lambda = \boldsymbol{y}_* + \boldsymbol{e} \in \mathbb{R}^\ell_+$, then $\boldsymbol{p}\boldsymbol{y}_* \leq \boldsymbol{p} \int_A (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda$. Hence we have proved that $Q_{\boldsymbol{e}} \neq \emptyset$ for any $\boldsymbol{e} \in E$ such that $\boldsymbol{y}_* + \boldsymbol{e} \in \mathbb{R}^\ell_+$. However, if $\boldsymbol{e} \leq \boldsymbol{e}'$, then $Q_{\boldsymbol{e}} \supset Q_{\boldsymbol{e}'}$. Thus $Q_{\boldsymbol{e}} \neq \emptyset$ for all $\boldsymbol{e} \in E$. Let $\boldsymbol{b} = (\epsilon, 1 \dots 1) \in E$. Then $Q_{\boldsymbol{e}} \subset Q_{\boldsymbol{b}}$ for all $\boldsymbol{e} \in E$.

For any finite family $\{\boldsymbol{e}_1 ... \boldsymbol{e}_n\}$ of E, define $\tilde{\boldsymbol{e}} = (\tilde{e}^t)$ by $\tilde{e}^t = max\{e^t_1 \dots e^t_n\}$. Then $\cap_{i=1}^n Q_{\boldsymbol{e}_i} \supset Q_{\tilde{\boldsymbol{e}}} \neq \emptyset$.

Let $\boldsymbol{q} \in \cap_{\boldsymbol{e} \in E} Q_{\boldsymbol{e}}$. Obviously we have $\boldsymbol{p}\boldsymbol{y}_* + \boldsymbol{q}\boldsymbol{z}_* = 0$. Now suppose that $\boldsymbol{z}_* \leq \boldsymbol{z} < \boldsymbol{0}$, and $(\boldsymbol{y}, \boldsymbol{z}) \in Y$. There exists an $\boldsymbol{e} \in E$ such that $\boldsymbol{y} \geq -B\|\boldsymbol{z}\|\boldsymbol{e}_*$. Since $\boldsymbol{q} \in Q_{\boldsymbol{e}}$, it follows that $\boldsymbol{p}\boldsymbol{y} + \boldsymbol{q}\boldsymbol{z} \leq 0$. For $\boldsymbol{z} = \boldsymbol{0}$, we have $\boldsymbol{p}\boldsymbol{y} \leq 0$, since $Y(\boldsymbol{0}) = \mathbb{R}^\ell_-$. If $z^s_* = 0$ for some $s = 1 \dots m$, the same arguments as above also applies by restricting $Q_{\boldsymbol{e}}$ by the set $Q'_{\boldsymbol{e}}$ of $\boldsymbol{q}$'s whose components corresponding to zero components of $\boldsymbol{z}_*$ are zero. It is sufficient to note that $z^s_* = 0$ implies that $\boldsymbol{p} \int_A (\boldsymbol{x}(a) - \boldsymbol{e}(a))d\lambda = 0$ and we can set $\zeta^s = q^s = 0$ in this case, we should set $B = m^{-1} \left(max\{-z^s_* | \quad - z^s_* > 0\}\right)^{-1}$. Lemma 3 then applies and the proof is complete. ∎

4 Concluding Remarks

1. Oddou (1976) constructed an example of a distributive production economy with the (atomless) measure space of consumers in which there exists an allocation which belongs to the core but fails to be a social equilibrium (see Oddou (1976, p.277)). Therefore if the distributive production sets are present in the economy, the core is generally larger than the set of social equilibria, hence "the core equivalence theorem (Aumann(1964) and Hildenbrand (1974))" does not hold. On account of this result (and from the title of the Scarf (1986)'s original paper), one might feel that the concept of social equilibrium is merely a tool for proving the existence of the core in economies with distributive production sets. The following statement of Scarf himself (Scarf (1986, Preface, p.406) would support this impression;

> "... In publishing this paper so many years after its writing, I am offering a public argument for my reluctantly acquired feeling that a replacement for the Walrasian model, incorporating economies of scale production, cannot be based on the concepts of cooperative game theory. In order to obtain such a theory we must have recourse either to considerations of imperfect competition and non-cooperative game theory (Hart (1982)), to non-strategic equilibrium concepts (Brown and Heal (1983, 1985), or to the study of indivisibility in production (Scarf (1981a, 1981b, 1984))."

In the above statement, Scarf seems to deny the significance of even the core itself in the distributive production economies. We would like to claim, on the contrary, the social equilibrium, when it was under consideration with the relationship between the core, is indeed more than just a device for the investigation of the core.

First of all, the social equilibrium is always in the core, hence it is unconditionally Pareto optimal, which is on the same footing as the competitive equilibrium being always Pareto optimal. Second, it is in a "stable" state in the sense that any non-null coalitions do not block it. Note that the marginal cost pricing equilibrium (e.g., Brown and Heal (1983, 1985) and Brown (1991) for general reference) or the monopolistically competitive equilibrium with large fixed costs (e.g., Dehez, Dreze and Suzuki (2002) and Suzuki (2009, Chapter 6) for general reference) do not have either of these properties.

Moreover, we have shown that the social equilibrium is sufficiently robust to include indivisible commodities naturally and that the dispersed endowment distribution assumption is sufficient for the existence of the social equilibrium (Theorem 1), which is in contrast to the competitive equilibrium of coalition production economies with convex production sets.

On account of these nice characters which in other equilibrium concepts of non-convex production economies or the convex (competitive) production economies with the continuum of consumers fail, it seems fair to say that the social equilibrium invented by Scarf should be taken seriously and interesting for its own sake.

2. We assumed that the space of the consumers is the unit interval $[0,1]$ of $\mathbb{R}$ with the Lebesgue measure λ. We do not think that this assumption is very restrictive, since atomless probability spaces which are not isomorphic to $([0,1],\lambda)$ are considered to be rather pathological (see Halmos et al (1942)). Indeed, this assumption was not used in the proof of Theorem 1 at all, and the proof of Theorem 2 can be generalized without any changes to the case where the space of consumers is an atomless probability space on a compact metric space.

3. We can prove Theorem 2 dropping the assumption (DD). Although this "Theorem" is logically correct, we should say that it holds in vain. Since there exist indivisible commodities in our economy, the core will be empty from the outset if we do not impose the dispersed assumption. Therefore the demonstration of the distributiveness of the production set has to be performed within the class of economies which satisfy the condition (DD). It is obvious that Theorem 1 can be easily extend to the case of the general non-convex consumption sets by invoking the dispersed condition of Yamazaki (1978). On the other hand, the extension of Theorem 2 to the case of general non-convex closed consumption sets, however, is an open problem.

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