Kenjiro Ara Meets Frank Ramsey: Unbalanced Growth in A Neoclassical Optimal Growth Model

Harutaka Takahashi[1]

Abstract

We will study the unbalanced growth in a neoclassical two-sector optimal growth model with sector specific technical progress, which is an optimal growth version of Ara (1969), and will demonstrate the existence and the saddle-point stability of the optimal steady state (OSS) in the efficiency-units. By so doing the efficiency-units OSS paths will exhibit the unbalanced growth in terms of the original-units; each sector will grow at its own growth rate. Furthermore the growth rate of the aggregated output (GDP) will converges to the one of the sector with the higher technical progress.

JEL Classification:O14,O21,O24,O41

1．INTRODUCTION

We have witnessed a recent resurgence of an interest on growth and structural change. In fact, the industry-based empirical studies across countries clearly have shown that growth in an individual industry's per capita capital stock and output grow at industry's own growth rate, which is closely related to its technical progress measured by total factor productivity (TFP) of the industry. For example, per capita capital stock and output of the agriculture industry grow at 5% per annum along its own steady-state, whereas they grow at 10% annually in the manufacturing industry, also paralleling the industry's steady state. Syverson (2011) has recently reviewed these arguments discussed above. Let us refer this phenomenon as **“unbalanced growth among**

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industries.”

The attempt to understand this phenomenon has generated a strong theoretical demand for constructing a multi-sector growth model, yet very little progress has been made so far with the only exception of Baumol (1967) and Ara (1969). Baumol (1967) has set up an unbalanced growth model and demonstrated that if resources are shifting from progressive sector like electrical machinery sector towards sectors where productivity is growing relatively slowly like service sector, the aggregate productivity growth rate will slow down. Based on this theoretical observation, he has concluded a very pessimistic result that costs of service industry; costs of education, fine arts and government services will increase forever. This phenomenon is often referred to as “Baumol’s cost disease.” On the other hand, Ara (1969) set up the Uzawa-type two-sector growth model with the Cobb-Douglas production functions with constant-returns to scale and proved that each sector’s growth rate will eventually converge to the growth rate determined by the sector-specific technical progress.

Recent exceptions are Echevarria (1997), Kongsamut, Rebelo and Xie (2001), Acemoglu and Guerrieri (2008) and Ngai and Pissariadis (2011). Setting up an optimal growth model with three sectors: primary, manufacturing and service, Echevarria (1997) has applied a numerical analysis to solve the model. Kongsamut, Rebelo and Xie (2001) have constructed the similar model to the one of Echevarria (1997), while they have investigated the model under a much stronger assumption than her: each sector produces goods with the same technology. In other words, they also assume a one-commodity economy. On the other hand, Acemoglu and Guerrieri (2008) have studied the model with two physically differentiated intermediate-goods sectors and single final-good sector. Note that the last two models will share a common character: one final good economy except Echevarria (1997). Ngai and Pissariadis (2011) has set up the multi-sector optimal growth model with the capital good and demonstrated that contrast to Baumol’s claim, the economy’s growth rate is not on an indefinitely declining trend.

It is important to note that all the analytical models mentioned above share a common defect; since they assume the same production functions among sectors, except sector specific exogenous technical progress, their models would be identified as one-commodity economy. Because of this property, they could aggregate sector’s output even in a transition process. Contrast to their model, we will study a two-commodity economy in this paper. We assume that each good is produced with a different technology; consumption-goods and capital-goods are completely differentiated physical characters. As I will demonstrate later, this feature of the model will make the characteristics of the model far more complicated.

In Section 1, I will set up a similar two-sector optimal growth model with the general neo-classical production functions to Ara's two-sector growth model which will be discussed in Appendix A, in which each industry exhibits the Hicks-neutral technical progress with an industry specific rate. Then I will study the model under the optimal growth setting, which shows a sharp contrast to Ara's analysis. In Section 2, I will rewrite the original model into a per capita efficiency unit model. Then, I will transform the efficiency unit model into a reduced form model, after which the method developed by Baierl, Nishimura and Yano (1998) will be applied to show the existence of unbalanced growth. In Section 3, the saddle-point stability of unbalanced growth will be also demonstrated.

2. MODEL AND ASSUMPTIONS

We will begin with listing up the notation:

r : a subjective rate of discount,

$C(t) \in \mathbb{R}_+$: the total good consumed at t,

$c(t) \in \mathbb{R}_+$: $C(t)/L(t) \in \mathbb{R}_+$,

$Y(t) \in \mathbb{R}_+$: the tth period capital output of the capital good sector,

$K(t) \in \mathbb{R}_+$: the total capital good at t,

$K(0) \in \mathbb{R}_+$: the initial total capital good,

$K_i(t) \in \mathbb{R}_+$: the tth period capital stock of the ith sector,

$L(t) \in \mathbb{R}_+$: the total labor input at t,

$L(0) \in \mathbb{R}_+$: the initial total labor input,

$L_i(t) \in \mathbb{R}_+$: the tth labor input of the ith sector,

δ : the depreciation rate,

$A_i(t)$: the Hicks neutral technical-progress of the ith sector,

where $i=0$ and $i=1$ indicate the consumption good sector and the capital good sector respectively.

We will make the following two assumptions on the model:

Assumption 1.

1) The utility function u(・) is defined on $\mathbb{R}_{++}$ as the following:

$$u(C(t)) = c(t) = C(t)/L(t)\ (>0\ for\ t \geq 0).$$

2) $L(t) = (1+g)^t L(0)$, where g is a rate of population growth.

Assumption 2.

1) All the goods are produced with the following Cobb-Douglass production functions with the Hicks-neutral technical progress:

$$C(t) = A_0(t)K_0(t)^{\alpha_1} L_0(t)^{\alpha_2}\ and\ Y(t) = A_1(t)K_1(t)^{\beta_1} L_1(t)^{\beta_2}$$

$$where\ \alpha_1 + \alpha_2 = 1\ and\ \beta_1 + \beta_2 = 1.$$

2) $A_i(t) = (1+a_i)^t A_i(0)\ (i=0,1)$, where a_i is a rate of output-augmented (the Hicks-neutral) technical-progress of the i th sector and given as $0 < a_i < 1$.

Note that 2) of Assumption 2 implies that the sector specific TFP is measured by the sector specific output-augmented technical progress (the Hicks-neutral technical progress), which is externally given.

Before setting up the model, we will divide all the variables by $A_i(t)L(t)$ and will transform the original variables into per -capita efficiency-units variables. Now let us define the following normalized variables:

$$\tilde{y}(t) = \frac{Y(t)}{A_1(t)L(t)},\ \tilde{c}(t) = \frac{C(t)}{A_0(t)L(t)},\ k_1(t) = \frac{K_1(t)}{L(t)},$$

$$k_0(t) = \frac{K_0(t)}{L(t)},\ \ell_1(t) = \frac{L_1(t)}{L(t)},\ \ell_0(t) = \frac{L_0(t)}{L(t)}.$$

Firstly, let us transform the both sector's production functions into the efficiency-units ones as follows; dividing both sides by $A_1(t)L(t)$, we will yield

$$\tilde{c}(t) = f^0(k_0(t), \ell_0(t)) = k_0(t)^{\alpha_1} \ell_0(t)^{\alpha_2}\ and\ \tilde{y}(t) = f^1(k_1(t), \ell_1(t)) = k_1(t)^{\beta_1} \ell_1(t)^{\beta_2}.$$

The next step is to derive the efficiency-units production possibility frontier (PPF) as shown in Lemma 1 :

Lemma 1. *The* ***efficiency-units production possibility frontier (the efficiency PPF*** *for short)* *:* $\tilde{c} = T\left(\tilde{y}, k\right)$ *is explicitly calculated as follows :*

$$\tilde{c} = T\left(\tilde{y}, k\right) = \left[\frac{\alpha_2\beta_1}{\Delta\left(k, \tilde{y}\right)}\right]^{\alpha_2}\left[k - e\left(k, \tilde{y}\right)\right] \tag{1}$$

, where $\Delta\left(k, \tilde{y}\right) = \alpha_2\beta_1 k + (\alpha_1\beta_2 - \alpha_2\beta_1)e\left(k, \tilde{y}\right)$ *and* $e\left(k, \tilde{y}\right)$ *is the function obtained by solving the following equation with respect to* k_1 *:*

$$\left(\alpha_1\beta_2\right)^{\beta_2} k_1 = \tilde{y}\left[\alpha_2\beta_1 k + (\alpha_1\beta_2 - \alpha_2\beta_1)k_1\right]^{\beta_2}.$$

Proof. We will apply the numerical method studied by Baierl, Nishimura and Yano (1998). Under Assumption 2, let us consider the problem (*) where the time index is dropped for simplicity :

$$(*)\ Max \quad \tilde{c} = k_0^{\alpha_1}\ell_0^{\alpha_2} \quad s.t.\ \tilde{y} = k_1^{\beta_1}\ell_1^{\beta_2},\ \ \ell_0 + \ell_1 = 1\ and\ k_0 + k_1 = k.$$

The profit-maximization of both sectors will yield the following first order conditions :

$$\frac{r}{w} = \frac{\alpha_1\ell_1}{\alpha_2 k_1} = \frac{\beta_1\ell_2}{\beta_2 k_2}.$$

Solving the above equation with respect to ℓ_1 and substituting into $\ell_0 + \ell_1 = 1$, we have

$$\ell_1 = \frac{\alpha_1\beta_2 k_1}{\alpha_2\beta_1 k + (\alpha_1\beta_2 - \alpha_2\beta_1)k_1}. \tag{2}$$

Furthermore, substituting (2) into $\tilde{y}(t) = k_1(t)^{\beta_1}\ell_1(t)^{\beta_2}$ yields

$$k_1\,(\alpha_1\beta_2)^{\beta_2} = \tilde{y}\left[\alpha_2\beta_1 k + (\alpha_1\beta_2 - \alpha_2\beta_1)k_1\right]^{\beta_2} = \tilde{y}\left[\Delta(k, \tilde{y})\right]^{\beta_2}, \tag{3}$$

where $\Delta\left(k, \tilde{y}\right) \equiv \alpha_2\beta_1 k + (\alpha_1\beta_2 - \alpha_2\beta_1)k_1$.

Solving (3) with respect to k_1 , we will obtain

$$k_1 = e\left(k, \tilde{y}\right). \tag{4}$$

Then substituting (2) and (3) into $\tilde{c} = k_0^{\alpha_1}\ell_0^{\alpha_2}$ and after some manipulations, we yield the following efficiency PPF eventually :

$$\tilde{c} = \left(\frac{\alpha_2\beta_1}{\Delta\left(k, \tilde{y}\right)}\right)^{\alpha_2}\left(k - e\left(k, \tilde{y}\right)\right).$$

■

Remark 1. *Note that the derived efficiency-units PPF is constructed such that the original-units PPF at each period will be pulled back to the corresponding efficiency-units PPF by discounting with each sector's rate of TFP growth ($A_i(t)$) as depicted by Figure 1, below: In Figure1, four PPF curves of t-th and (t+1)-th periods are drawn. The original-units PPF and the corresponding efficient-units PPF curves are drawn. The efficient-units PPF curves at t-th period will be constructed by discounting back the original-unit PPF at each sector's TFP growth rate along the each axis. The efficiency-units PPF at (t+1) th period will be obtained by applying the same procedures to the original-units PPF at (t+1) th period.*

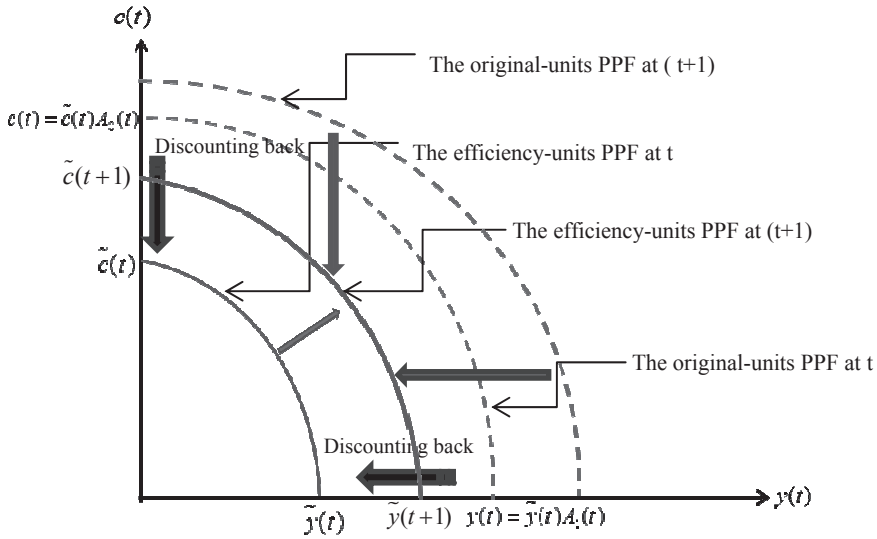

Figure 1. The Efficiency-units PPF

Now let us construct the following optimal growth model in terms of the per-capita efficiency-units:

-The Per-capita Efficiency-unit Optimal Growth Model-

$$Max \sum_{t=0}^{\infty} \rho^t \tilde{c}(t)$$

$$s.t.\ k(0)=\bar{k},$$

$$\tilde{c}(t)=T(\tilde{y}(t),k(t)) \quad (t=0,1,\cdots)$$

$$\tilde{y}(t)+(1-\delta)k(t)-(1+g)k(t+1)=0 \quad (t=0,1,\cdots). \tag{5}$$

The objective function in terms of per-capita term is derived as follows: Let us rewrite the consumption in terms of efficiency-units.

$$\tilde{c}(t) = \frac{C(t)}{A_0(t)L(t)} = \frac{c(t)}{(1+a_0)^t A_0(0)} = \frac{c(t)}{(1+a_0)^t},$$

where I assume that $A_0(0) = 1$ for simplicity.

Then the infinite discounted sum of consumptions[2] will be rewritten in terms of efficiency-units as follows :

$$\sum_{t=0}^{\infty}\left[\frac{(1+a_0)}{(1+r)}\right]^t \tilde{c}(t) = \sum_{t=0}^{\infty}\rho^t \tilde{c}(t)$$

where

$$\rho = \frac{1+a_0}{1+r}.$$

We will make the following additional assumption here:

Assumption3. $A_0(0) = 1$ *and* $g < a_0 < r$.

The accumulation equation (5) is constructed based on the efficiency-units PPF as explained in Remark 3, where

$$\frac{K(t)}{L(t)} = k(t) \text{ and } \frac{K(t+1)}{L(t)} = \frac{(1+g)K(t+1)}{(1+g)L(t)} = (1+g)k(t+1).$$

The following important remark is in order now :

Remark 2. *It is important to note that the accumulation equation (5) will be directly derived from rewriting the following efficient-units accumulation equation constructed based on the efficiency-units PPF by dividing both sides with* $L(t)$ *:*

$$\tilde{Y}(t) + (1-\delta)K(t) - K(t+1) = 0.$$

Observing Figure 5, the TFP growth effect will be annihilated by discounting the original-units variables with each sector's TFP growth rate, but the efficiency PPF could still expand outward because of the capital accumulation itself. Indeed, the equation (5) will exhibit this process.

If x and z indicate initial and terminal capital stocks respectively, the reduced form utility function $V(x,z)$ and the feasible set D will be defined as follows:

$$V(x,z) = u(T[(1+g)z - (1-\delta)x, x])$$

2 This type of objective function was originally used by Uzawa (1964).

and

$$D = \{(x,z) \in \mathbb{R}_+ \times \mathbb{R}_+ : T[(1+g)z-(1-\delta)x, x] \geq 0\},$$

where $x = k(t)$ *and* $z = k(t+1)$. Note that we eliminate the time index for simplicity.

Finally, the per-capita efficiency-units model will be summarized as the following standard reduced form model, which have been studied in detail by Scheinkman (1976) and McKenzie (1986).

-The Reduced Form Model-

$$(**) Max \sum_{t=0}^{\infty} \rho^t V(k(t), k(t+1)) \; s.t. \; (k(t), k(t+1)) \in D \; for \; t \geq 0 \; and \; k(0) = \bar{k}.$$

Also note that any interior optimal path must satisfy the following Euler equation, which exhibits an inter-temporal efficiency allocation condition:

$$V_z(k(t-1), k(t)) + \rho V_x(k(t), k(t+1)) = 0 \quad for\ all\ t \geq 0 \tag{6}$$

, where the partial derivatives mean that

$$V_x(k(t), k(t+1)) = \frac{\partial V(k(t), k(t+1))}{\partial k(t)} \; and \; V_z(k(t-1), k(t)) = \frac{\partial V(k(t-1), k(t))}{\partial k(t)}.$$

Note that under the differentiability assumptions, due to the envelope theorem, all the prices will be obtained as the following relations:

$$q = \frac{d\tilde{c}}{d\tilde{c}} = 1, \; p = -q\frac{\partial T(\tilde{y}, k)}{\partial \tilde{y}}, \; w = q\frac{\partial T(\tilde{y}, k)}{\partial k}, \; and \; w_0 = q\tilde{c} + p\tilde{y} - wk$$

, where we normalize the price of consumption good as 1.

Now we will define the optimal steady state as follows :

Definition. An optimal steady state path (OSS) k^{ρ} is an optimal path which solves the reduced-form model (**) and satisfies $k^{\rho} = k(t) = k(t+1)$ *for all* $t \geq 0$.

3. UNBALANCED GROWTH

In this section, based on the Cobb-Douglas technology, we will calculate the optimal steady state numerically.

3. 1. Existence of the OSS

Using the accumulation equation and let us define

$$V(x,z) = T\left[(1+g)z-(1-\delta)x, x\right] \; where \; x = k(t), \; z = k(t+1).$$

Then, the Euler equation will be derived as follows :

$$(1+g)\frac{\partial T}{\partial \tilde{y}}+\left(\frac{\rho}{1+g}\right)\left[-(1-\delta)\frac{\partial T}{\partial \tilde{y}}+\frac{\partial T}{\partial k}\right]=0. \tag{7}$$

Evaluating (7) along the OSS yields,

$$\left(\frac{\rho}{1+g}\right)\left[p^{\rho}(1-\delta)+r^{\rho}\right]=p^{\rho} \tag{8}$$

From (2),

$$k_1^{\rho}=\beta_1\tilde{y}^{\rho}\left(\frac{r^{\rho}}{p^{\rho}}\right). \tag{9}$$

From (8),

$$\left(\frac{p^{\rho}}{r^{\rho}}\right)=\frac{\rho}{(1+g)-\rho(1-\delta)}\equiv m. \tag{10}$$

Eliminating $\left(\frac{p^{\rho}}{r^{\rho}}\right)$ from (9) and (10) yields,

$$k_1^{\rho}=\beta_1 m\tilde{y}^{\rho} \tag{11}$$

Furthermore, substituting again (11) into (3) and using the facts that $\tilde{y}^{\rho}=(g+\delta)k^{\rho}$, we have

$$(\alpha_1\beta_2)^{\beta_2}(\beta_1 m)=\beta_1^{\beta_2}\left[\alpha_2+(\alpha_1\beta_2-\alpha_2\beta_1)m(g+\delta)\right]^{\beta_2}\left(k^{\rho}\right)^{\beta_2} \tag{12}$$

Solving (12) with respect to k^{ρ}, we have proved the following result.

Proposition 1. *The OSS is numerically obtained as follows :*

$$k^{\rho}=\frac{(\alpha_1\beta_2)(\beta_1 m)^{\frac{1}{\beta_2}}}{\beta_1\left[\alpha_2+(\alpha_1\beta_2-\alpha_2\beta_1)m(g+\delta)\right]}\quad where\ m=\frac{\rho}{(1+g)-\rho(1-\delta)}.$$

3. 2. Unbalanced growth

Due to Proposition 1 and the accumulation equation (5), $\tilde{y}^{\rho}=(g+\delta)k^{\rho}$ holds. Thus the output of the capital good sector at the original-units value: $\tilde{y}^{\rho}(t)$ can be expressed as

$$\tilde{y}^{\rho}(t)=(1+a_1)^t A_1(0)\tilde{y}^{\rho}.$$

This means that the per-capita output of capital good sector grows at its TFP growth rate along the OSS path.

Furthermore, since $\tilde{c}^{\rho} = T\left(\tilde{y}^{\rho}, k^{\rho}\right)$ holds, it follows that $\tilde{c}^{\rho}(t) = (1+a_0)^t A_0(0)\tilde{c}^{\rho}$. Thus the per-capita output of consumption good sector also grows at its TFP growth rate.

The above result is summarized as the following corollary:

Corollary. *The optimal consumption and capital output steady state paths, denoted by* $c(t)^{\rho}$ *and* $y(t)^{\rho}$ *respectively, are growing at its own TFP rate:* a_0 *and* a_1 *respectively.*

This result can be illustrated as Figure 6, where each sector's production function will shift at its own TFP rate. Note that $p^{\rho} = -\dfrac{\partial T(\tilde{y}^{\rho}, k^{\rho})}{\partial \tilde{y}}$ *and* $w^{\rho} = \dfrac{\partial T(\tilde{y}^{\rho}, k^{\rho})}{\partial k}$. Instead of using those expressions, to investigate the prices, it is more convenient to rewrite the prices in terms of the conventional rental-wage ratio used by Uzawa (1964). Note that the following relations concerned with the rental-wage ratio, often denoted by ω will hold:

$$\omega(t) = \frac{w(t)}{w_0(t)} = \frac{A_0(t)\left[f^0\left(\overline{k_0}(t),1\right) - \overline{k_0}(t)\dfrac{\partial f^0}{\partial \overline{k_0}(t)}\right]}{A_0(t)\dfrac{\partial f^0}{\partial \overline{k_0}(t)}} = \frac{A_1(t)\left[f^1\left(\overline{k_0}(t),1\right) - \overline{k_1}(t)\dfrac{\partial f^1}{\partial \overline{k_1}(t)}\right]}{A_1(t)\dfrac{\partial f^1}{\partial \overline{k_1}(t)}}$$

$$, where \quad \overline{k_0}(t) = \frac{k_0(t)}{\ell_0(t)} \; and \; \overline{k_1}(t) = \frac{k_1(t)}{\ell_1(t)}.$$

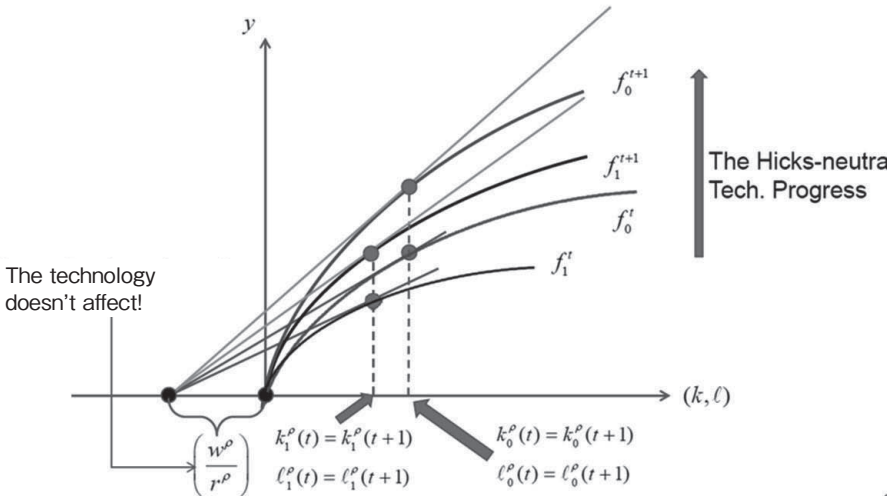

Figure 2. Properties of the Unbalanced Growth

Based on these relations, due to the fact that k^{ρ} is constant along the OSS, ω^{ρ} is also constant. Since the relative price of capital good: $p(t)$ is a function of $\omega(t)$, it must be also constant along the OSS.

Now we can define the Gross Domestic Products on the OSS (the OSS-GDP for short) as follows:

$$c^{\rho}(t) + p^{\rho} y^{\rho}(t) = A_0(t)\tilde{c}^{\rho} + A_1(t) p^{\rho} \tilde{y}^{\rho}.$$

Since the growth rate of the OSS-GDP is expressed as the weighted average of the growth rate of both sectors :

$$\textit{The growth rate of the OSS} - GDP = \left(\frac{A_0(t)\tilde{c}^{\rho}}{A_0(t)\tilde{c}^{\rho} + A_1(t) p^{\rho} \tilde{y}^{\rho}} \right) a_0 + \left(\frac{A_1(t) p^{\rho} \tilde{y}^{\rho}}{A_0(t)\tilde{c}^{\rho} + A_1(t) p^{\rho} \tilde{y}^{\rho}} \right) a_1$$

Thus the GDP grows at a certain constant rate of growth calculated as the weighted sum of the growth rates of both sectors. Based on the above relation, we will easily demonstrate the following proposition:

Proposition 2. If $a_0 > a_1 (a_0 < a_1)$, then the OSS − GDP growth rate will converge to $a_0 (a_1)$.

Note that this proposition exhibits a sharp contrast to the result referred to as "Baumol's cost disease": the GDP growth rate will converges to that of the stagnated sector. On the contrary, Proposition 2 implies that the growth rate of the GDP will converges to that of the non-stagnated sector. Thus we have proved the anti-Baumol thesis.

4. LOCAL STABILITY

In this section we will show the local stability in the sense of saddle-point stability. By so doing, we demonstrate that the corollary of Proposition 2 will take place at the limit.

4. 1. Capital intensity

Let us first define the capital intensity of a sector.

Definition. (capital intensity) *When* $\alpha_1 / \alpha_2 > \beta_1 / \beta_2 (\alpha_1 / \alpha_2 < \beta_1 / \beta_2)$ *holds , we may say that the consumption good sector is* ***capital intensive (labor intensive)*** *in comparison with the capital good sector.*

As we have examined in Takahashi, Mashiyama and Sakagami (2011), we have presented firm

evidence such that in any OECD countries the consumption good sector is capital intensive. However, we will study both cases here.

4. 2. Local stability

By expanding the Euler equation around k^{ρ}, the following characteristic equation will be derived:

$$\rho^{*}V_{xz}^{\rho}\lambda^{2}+(V_{zz}^{\rho}+\rho^{*}V_{xx}^{\rho})\lambda+V_{zx}^{\rho}=0 \quad where\ \rho^{*}=\frac{\rho}{1+g},$$

If $V_{xz}^{\rho}\neq 0$, then we will yield

$$\rho^{*}\lambda^{2}+\frac{(V_{zz}^{\rho}+\rho^{*}V_{xx}^{\rho})}{V_{xz}^{\rho}}\lambda+\frac{V_{zx}^{\rho}}{V_{xz}^{\rho}}=0 \ . \tag{13}$$

The local stability will be determined by this equation. Let us solve (13) numerically.

Lemma 2[3]**.** *The characteristic equation (13) has the following two roots :*

$$\begin{cases} \lambda=\dfrac{\alpha_2\left[1-m(1-\delta)\right]+m(1-\delta)(\alpha_2-\beta_2)}{m(\alpha_2-\beta_2)} \\ \dfrac{1}{\rho\lambda}=\dfrac{1}{\rho}\left\{\dfrac{m(\alpha_2-\beta_2)}{\alpha_2\left[1-m(1-\delta)\right]+m(1-\delta)(\alpha_2-\beta_2)}\right\} \end{cases},$$

In other words, λ is a root, then $\dfrac{1}{\rho\lambda}$ is also the root of the characteristic equation (13).

Proof. Linearizing the Euler equation (6) around the OSS yields the following well-known characteristic equation (13).

By the facts that $V_x=-(1-\delta)T_z+T_x$ *and* $V_z=T_z$, we will obtain the followings :

$$\begin{aligned} V_{xx}^{\rho}&=(1-\delta)^2T_{zz}^{\rho}-(1-\delta)T_{zx}^{\rho}-(1-\delta)T_{xz}^{\rho}+T_{xx}^{\rho} \\ &=(1-\delta)\left[(1-\delta)T_{zz}^{\rho}-T_{zx}^{\rho}\right]+\left[\frac{\Delta_x^{\rho}}{\Delta_z^{\rho}}-(1-\delta)\right]T_{xz}^{\rho}, \end{aligned} \tag{14}$$

$$V_{xz}^{\rho}=V_{zx}^{\rho}=-T_{zz}^{\rho}(1-\delta)+T_{xz}^{\rho}, \tag{15}$$

3 Levhari and Liviatan (1972) has proved this property in a general optimal growth model.

$$V_{zz}^{\rho} = T_{zz}^{\rho}, \tag{16}$$

where $\Delta_x^{\rho} = \alpha_2\beta_1 + (\alpha_1\beta_2 - \alpha_2\beta_1)\dfrac{\partial k_1^{\rho}}{\partial x}$ *and* $\Delta_z^{\rho} = (\alpha_1\beta_2 - \alpha_2\beta_1)\dfrac{\partial k_1^{\rho}}{\partial \tilde{y}}$.

From (3),

$$\frac{\partial k_1^{\rho}}{\partial x} = \frac{\tilde{y}^{\rho}\beta_2(\Delta^{\rho})^{\beta_2-1}\Delta_x^{\rho}}{(\alpha_1\beta_2)^{\beta_2}} \text{ and } \frac{\partial k_1^{\rho}}{\partial \tilde{y}} = \frac{(\Delta^{\rho})^{\beta_2} + \tilde{y}^{\rho}\beta_2(\Delta^{\rho})^{\beta_2-1}\Delta_z^{\rho}}{(\alpha_1\beta_2)^{\beta_2}}.$$

Then we may show :

$$T_{xz}^{\rho} = -\frac{\alpha_1}{\beta_1}\left(\frac{\Delta^{\rho}}{\alpha_2\beta_1}\right)^{-\alpha_2-1}\Delta_z^{\rho}, \quad T_{xx}^{\rho} = T_{xz}^{\rho}\frac{\Delta_x^{\rho}}{\Delta_z^{\rho}}, \quad T_{zz}^{\rho} = -T_{xz}^{\rho}\frac{(\alpha_2-\beta_2)m}{\alpha_2}. \tag{17}$$

Some manipulations will yield

$$\frac{\Delta_x^{\rho}}{\Delta_z^{\rho}} = \frac{\alpha_2}{(\alpha_1\beta_2 - \alpha_2\beta_1)m} = \frac{\alpha_2}{-(\alpha_2-\beta_2)m} \tag{18}$$

Substituting (17) and (18) into the equations (9) through (11) will yield

$$\frac{\rho^* V_{xx}^{\rho}}{V_{xz}^{\rho}} = \rho^*\left[\frac{(1-\delta)m(\alpha_2-\beta_2)+\alpha_2[1-m(1-\delta)]}{m(\beta_2-\alpha_2)}\right]$$

, and

$$\frac{V_{zx}^{\rho}}{V_{xz}^{\rho}} = \frac{m(\beta_2-\alpha_2)}{(1-\delta)m(\alpha_2-\beta_2)+\alpha_2[1-m(1-\delta)]}.$$

Thus the Euler equation will be eventually rewritten as follows :

$$\rho^*\lambda^2 + \left\{\rho^*\left[\frac{(1-\delta)m(\alpha_2-\beta_2)+\alpha_2[1-m(1-\delta)]}{m(\beta_2-\alpha_2)}\right] + \left[\frac{m(\beta_2-\alpha_2)}{(1-\delta)m(\alpha_2-\beta_2)+\alpha_2[1-m(1-\delta)]}\right]\right\}\lambda + 1 = 0.$$

Thus we obtain the two roots of the equation (13).■

Based on Lemma 2, we can show the following proposition :

Proposition 3 . *The following two properties hold :*

1) *If the consumption sector is capital intensive, and* $\delta + 2g < 1$[4] *, then there exists* $\rho' > 0$ *such that* $\rho \in [\rho', 1)$ *implies that* $-1 < \lambda < 1$.

2) If the capital good sector is capital intensive, then there exists $\rho'' > 0$ such that

4 From the Penn World Table, we may observe that $\delta \cong 0.25$ and $g \cong 0.25$ at most. Therefore this condition could be justified.

$\rho \in [\rho'',1)$ *implies that* $0 < \lambda < 1$.

Proof. Based on Lemma2, let us define the following functions :

$$\lambda(\rho) = \frac{\alpha_2[1-m(\rho)(1-\delta)]}{m(\rho)(\alpha_2-\beta_2)} + (1-\delta) = \left(\frac{\alpha_2}{\alpha_2-\beta_2}\right)\left[\frac{1}{m(\rho)} - (1-\delta)\right] + (1-\delta) \tag{14}$$

$$where\ m(\rho) = \frac{\rho}{(1+g)-\rho(1-\delta)}.$$

Differentiating (14) yields,

$$\lambda'(\rho) = \left(\frac{\alpha_2}{\alpha_2-\beta_2}\right)\left(\frac{-m'(\rho)}{m^2(\rho)}\right) . \tag{15}$$

and

$$m'(\rho) = \frac{(1+g)}{[(1+g)-\rho(1-\delta)]^2} > 0 , \tag{16}$$

The second term of the right hand of (15) is positive.

Also note,

$$\lambda(1) = \left(\frac{1}{1-\left(\frac{\beta_2}{\alpha_2}\right)}\right)[g+2\delta-1] + (1-\delta) . \tag{17}$$

I will prove the proposition in two cases.

Case 1: The consumption sector is capital intensive.

In this case, $\alpha_2 < \beta_2$ holds. Then we have the following result

$$\lambda'(\rho) < 0 \ and\ \lambda(1) > 0.$$

However, the sing of $\lambda(\rho)$ is undetermined. $\lambda(1) > 0$ comes from the fact that $g + 2\delta < 1$.

Thus there exists $\rho' > 0$ such that

$$\rho \in [\rho',1)\ implies\ that\ -1 < \lambda < 1.$$

Case 2: The consumption sector is capital intensive.

Due to the definition, $\alpha_2 > \beta_2$ holds. From (15) through (17), it follows that $\lambda(\rho) > 0, \lambda'(\rho) < 0\ and\ \lambda(1) > g + \delta$. The last result comes from the fact that

$$\frac{1}{1-\left(\frac{\beta_2}{\alpha_2}\right)} > 1.$$

Therefor there exists $\rho'' > 0$ such that $\rho \in [\rho'',1)$ *implies that* $0 < \lambda < 1$.

This completes the proof. ■

From Proposition 2, there exists a $\hat{\rho} > 0$ such that the characteristic equation (13) has one root with its absolute value less than one and the other characteristic root with its absolute value greater than one $for\ \rho \in [\hat{\rho}, 1)$. Thus we have demonstrated the local stability.

Proposition 4. *Under the both capital intensity condition*[5]*s, there exists a* $\hat{\rho} > 0$ *such that the optimal steady state* k^{ρ} $for\ \rho \in [\hat{\rho}, 1)$ *is locally stable in the sense of the saddle point .*

Corollary: *Each sector's optimal path in the neighborhood of the optimal steady state* k^{ρ} $for\ \rho \in [\hat{\rho}, 1)$ *will converge to its own optimal steady state :*

$$(\tilde{c}^{\rho}(t), \tilde{y}^{\rho}(t)) \to (\tilde{c}^{\rho}, \tilde{y}^{\rho})\ as\ t \to \infty.$$

It follows that in terms of the original-units variables, sector's per-capita capital stock and output eventually grow at the rate of sector's TFP growth:

$$c^{\rho}(t) = (1+a_0)^t \tilde{c}^{\rho}\ and\ y^{\rho}(t) = (1+a_1)^t \tilde{y}^{\rho}.$$

5. CONCLUTION

We have shown the existence of the unbalanced optimal steady state growth path and, by demonstrating the saddle-point stability, the optimal path of each sector will converge to its own unbalanced growth path with its TFP growth rate. Thus in the end, the sector with the higher TFP growth rate will dominate at the aggregated GDP growth rate. This result shows a sharp contrast to that of Baumol (1964).

Needless to say, the existence and the global stability of the efficiency OSS under the assumptions of general production functions among sectors should be left for a further research problem.

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5 As we have discussed in Section 2, the empirical supports the fact that the consumption-goods sector is capital intensive among OECD countries.

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Appendix A

Let us review the two-sector growth introduced by Ara (1969)[6] in terms of our model notation used in the model. Ara (1969) has set up a continuous-time Uzawa-type two-sector non-optimal growth model with the following Cobb-Douglas production functions:

$$\begin{cases} C(t) = A_0 e^{a_0} K_0(t)^{\alpha_1} L_0(t)^{\alpha_2} \\ \\ Y(t) = A_1 e^{a_1} K_1(t)^{\beta_1} L_1(t)^{\beta_2} \end{cases}$$

He also made the following two extra assumptions; (A) and (B) other than ours:

(A) Economic agents have a static expectation.

(B) Labor income is only consumed and rental income is solely saved.

Let us define $G(x) = (1/x)(dx/dt)$ and the long-run equilibrium will be defined as the economic situation in which $G(K_1(t)) = G(K_2(t)) = G(K(t)) = w^*$ holds, where w^* stands for the long-run equilibrium profit rate, which is stable. Then his model in the long-run will be eventually characterized by the following seven equations:

(i) $\quad G(C(t)) = \dfrac{a_0}{\alpha_2} + n,$

(ii) $\quad w^* = \dfrac{a_0}{\alpha_2} + n,$

(iii) $\quad G(K(t)) = \dfrac{a_0}{\alpha_2} + n,$

(iv) $\quad G(L(t)) = n,$

(v) $\quad G(Y(t)) = a_1 + \beta_1 \dfrac{a_0}{\alpha_2} + n,$

(vi) $\quad G(p(t)) = a_1 \left(\dfrac{a_1}{\beta_2} - \dfrac{a_0}{\alpha_2} \right),$

(vii) $\quad G(w_0(t)) = a_1 + \beta_1 \dfrac{a_0}{\alpha_2}.$

We may easily notice that his model is an unbalanced growth model, because $G(C(t)) - G(L_0(t)) = \dfrac{a_0}{\alpha_2} \neq G(Y(t)) - G(L_1(t)) = a_1 + \beta_1 \dfrac{a_0}{\alpha_2}$ holds. Therefore each sector's

6 Since Ara (1969) was written only in Japanese, very little attention to the paper has been made.

per-capita output grows at the sector-specific growth rate. However, note that his results show a sharp contrast to the one we obtained here in three points; firstly, we have proved that in his terms,

$$G(C(t)) - G(L_0(t)) = a_0 \neq G(Y(t)) - G(L_1(t)) = a_1 .$$

Secondly, we have shown that the long-run per capita capital total stocks are constant. On the contrary, from (*iii*) and (*iv*), $G(K(t)) - G(L(t)) = \frac{a_0}{\alpha_2}$ and the long-run per capita total capital is growing. Thirdly, in his model, all the prices grow as shown by (*vi*) and (*vii*). In our model those are constant. Of course these differences come from his special model setting and assumptions.