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The Ausubel Auction with the Receive Option of Over-Assignments*

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Abstract

The Ausubel auction, which is introduced by Ausubel (2004), is a dy-namic version of the Vickrey auction. Okamoto (2018) shows that in the original Ausubel auction, sincere-bidding by all bidders may not be an ex-post perfect equilibrium, and modifies the Ausubel auction, the *Ausubel auction with sequential rationing*, so that sincere-bidding by all bidders is always an ex-post perfect equilirium. In this paper, we propose another modification of the Ausubel auction, namely the *Ausubel auction with the receive option of over-assignments*. We also show that in this auction, sincere-bidding by all bidders is always an ex-post perfect equilibrium, which obtains the Vickrey outcome.

1. Introduction

Ausubel (2004) designs a dynamic auction for multiple homogeneous objects, called the *Ausubel auction*, which obtains the *Vickrey outcome* at a sincere-bidding equilibrium.[1] Okamoto (2018) shows that in the Ausubel auction, sincere-bidding by all bidders is always an *ex-post equilibrium*, but may not be an *ex-post perfect equilibrium*. The notion of ex-post perfect equilibrium requires that the strategy profile is an ex-post equilibrium at all subgames. In other words, if a strategy profile is not an ex-post perfect equilibrium, then some bidder has an in-centive to deviate at some subgame. Therefore, in the Ausubel auction, a bidder may not bid sincerely after making a bidding mistake. Okamoto (2018) modifies the Ausubel auction by introducing a new rationing rule, *sequential rationing*, so that

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1 The Vickrey outcome is an outcome of the Vickrey auction, which is introduced by Vickrey (1961). In the mechanism design literature, the Vickrey outcome is often called the VCG outcome. (Vickrey (1961), Clarke (1971), Groves (1973)) See, for example, Krishna (2009) for details.

sincere bidding by all bidders is always an ex-post perfect equilibrium.

Ausubel (2018) states that the optimality of sincere-bidding in dynamic auc-tions may be sensitive to the fine details of the rationing rule for tie-breaking. In the *dynamic* Ausubel auction with sequential rationing, all bidders have in-centives to bid sincerely even after making a mistake. However, in the *dynamic* Ausubel auction, under some rationing rules for tie-breaking, some bidder may have an incentive to underbidding after overbidding.[2] On the other hand, in the *static* Vickrey auction, under any rationing rule for tie-breaking, all bidders have strong incentives to bid sincerely, that is, sincere-bidding is a weakly dominant strategy for each bidder. There is only one round in static auctions, bidders have no chance to bid after making mistakes, and we do not have to consider bidders' incentives after that.

Seminal works investigate auction rules which always assign all the auctioned objects entirely according to a rationing rule for tie-breaking. In those auctions, bidders may be assigned in excess of their (last) bids without they are not asked whether they want those or not. In fact, in the Ausubel auction with sequential rationing, if a bidder overbids at some round, then the bidder may be over-assigned in the next round, that is, bidders could be punished for bidding mistakes even in the next round.

In this paper, we introduce a new auction rule, the *Ausubel auction with the receive option of over-assignments*, in which at each round, each bidder bids a quantity of objects and a signal as to whether or not the bidder receives the excess supply. In this auction, even if a bidder overbids at some round, the bidder is not be over-assigned in the next round by bidding the signal not to receive the excess supply. We then show that in this auction, sincere-bidding by all bidders is always an ex-post perfect equilibrium. Furthermore, in a sincere-bidding equilibrium path, this auction assigns all the auctioned objects entirely and obtains the Vickrey outcome.

This paper is organized as follows. In Section 2, we introduce definitions. In Section 3, we show our main result. In Section 4, we conclude our discussions.

2. Definitions

We almost follow notation of Ausubel (2004) and Okamoto (2018).

2-1. Bidders

A seller puts *M homogeneous divisible objects* for an auction. There is a finite set of *bidders* $N=$

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[2] For example, consider *proportional rationing*, which is introduced by Ausubel (1997). Okamoto (2018) gives an example such that under any rationing rule which satisfies mono-tonicity property, sincere bidding by all bidders is not an ex-post perfect equilibrium.

$\{1, 2, \ldots, n\}$ with $n \geq 2$. Each bidder $i \in N$ has a *consumption set* $X_i = [0, \lambda_i]$ with $0 < \lambda_i \leq M$ and a *valuation function* $U_i : X_i \to \mathbb{R}_+$. When a bidder $i \in N$ is assigned $x_i \in X_i$ and pays $y_i \in \mathbb{R}$, bidder i's utility is $U_i(x_i) - y_i$. For each $x_i \in X_i$, the value $U_i(x_i)$ can be calculated by the integral of corresponding *marginal value* $u_i : X_i \to \{0, 1, \ldots, \bar{u}\}$, so that

$$U_i(x_i) = \int_0^{x_i} u_i(q)\,dq$$

where $\bar{u}$ is a positive integer. We introduce the three assumptions on u_i such that (i) each u_i is a *weakly decreasing* function in X_i, (ii) $u_i(x_i)$ is an integer in $\{0, 1, \ldots, \bar{u}\}$ for all $x_i \in X_i$, and (iii) $\bar{u} < \infty$.

2-2. Auction rule

We shall offer the *Ausubel auction with the receive option over-assignments*. This auction is almost the same as the Ausubel auction, except for signals $(a_i)_{i \in N}$ as to whether or not the bidders receive the excess supply.

An auction proceeds by discrete time $\{0, 1, \ldots, T\}$ with $T < \bar{u}$. For each time $t \in \{0, 1, \ldots, T\}$, let the price $p^t = t$. An auction starts at $t = 0$, and it proceeds as follows.

$\boldsymbol{t = 0}$**:** Each bidder $i \in N$ simultaneously bids a quantity $x_i^0 \in X_i$ and a *signal* $a_i \in \{0, 1\}$ for *tie-breaking*. The signal $a_i^0 = 0$ means that i does not want to receive the excess supply, and $a_i^0 = 1$ means that i can receive the excess supply. If $\sum_{i \in M} x_i^0 \leq M$, the auction ends at $t = 0$ with assignment $(x_i^*)_{i \in N}$ such that

$$x_i^* = x_i^0 \quad \forall i \in N.$$

Otherwise, for each bidder $i \in N$, let

$$C_i^0 = \max\left\{0, M - \sum_{i \neq i} x_j^0\right\}$$

be bidder i's *cumulative clinches* at $t = 0$, and the auction continues to $t = 1$.

$\boldsymbol{t = s < T}$**:** The auctioneer announces full information of prior bids to each bidder. Each bidder $i \in N$ simultaneously bids a quantity x_i^s satisfying the bidding constraint

$$C_i^{s-1} \leq x_i^s \leq x_i^{s-1}.$$

and a singal $a_i^s \in \{0, 1\}$. If $\sum_{i \in N} x_i^s \leq M$, the auction ends at $t = s$ with an assignment x_i^* which is decided by the following way: Let $N_0 = \{i \in N : a_i = 0\}$ and $N_1 = \{i \in N : a_i = 1\}$. Then,

$$x_i^* = x_i^s \quad \forall i \in N_0$$

$$x_j^* = x_j^s + \min\left\{x_j^{s-1} - x_j^s,\ \frac{x_j^{s-1} - x_j^s}{\sum_{k\in N_1} x_k^{s-1} - x_k^s}\cdot\left(M - \sum_{i\in N} x_i^s\right)\right\} \quad \forall j\in N_1.$$

Otherwise, let $C_i^s = \max\left\{0,\ M - \sum_{j\neq i} x_i^s\right\}$ be bidder i's cumulative clinches, and the auction continues to $t = s+1$.

t**=*****T*****:** The auctioneer announces full information of prior bids to each bidder. Each bidder $i\in N$ simultaneously bids a quantity x_i^T satisfying the bidding constraint

$$C_i^{T-1} \leq x_i^T \leq x_i^{T-1}.$$

and a signal $a_i^T \in \{0, 1\}$. In any case, the auction ends, even when there is the excess demand. If $\sum_{i\in N} x_i^T > M$, an assignment $(x_i^*)_{i\in N}$ such that $\sum_{i\in N} x_i^* = M$ and

$$x_i^* \leq x_i^T \quad \forall i\in N.$$

Otherwise, similarly to the case ends at $t = s < T$, an assignment $(x_i^*)_{i\in N}$ is decided.

This auction process finishes in at most $T+1$ rounds. Let L be the *last round* of the auction game, i.e., $\sum_{i\in N} x_i^L \leq M$ or $L = T$. For each bidder $i\in N$, define cumulative clinches of the last time by i's assignment, $C_i^L = x_i^*$. Then, by this process, we obtain a vector of cumulative clinches $\left\{(C_i^t)_{i\in N}\right\}_{t=0}^L$. We then define the vector of current clinches $\left\{(c_i^t)_{i\in N}\right\}_{t=0}^L$ as follows: For each $i\in N$ and $t\geq 1$,

$$c_i^t = C_i^t - C_i^{t-1},$$

and $c_i^0 = C_i^0$.

For each bidder $i\in N$, the payment y_i is calculated as the bidder buys each current clinch c_i^t at each price p^t. Therefore, the *payment* is given by

$$y_i = \sum_{i=0}^{L} p^t c_i^t.$$

2-3. Histories and Strategies

At each time $t\in\{0, 1, ..., T\}$, a *history* is a vector of all prior bids to t

$$h^t = \left((x_1^s, a_1^s), (x_2^s, a_2^s), \ldots, (x_N^s, a_N^s)\right) \in \left(\times_{i\in N}(X_i \times \{0, 1\})\right)^{\{0, 1 \ldots t-1\}}$$

such that for each $i\in N$ and each $s\leq t-1$,

$$C_i^{s-1} \leq x_i^s \leq x_i^{s-1},$$

$$\sum_{j\in N} x_j^{t-2} > M.$$

Define the history of starting point $t=0$ by the *empty sequence*, $h^0=\emptyset$.

Let H^t be the set of histories at t, and $H\equiv \bigcup_{t=0}^{T+1} H^t$ be the set of all histories. A history

$$z^{t+1}=((x_1^s,a_1^s),\ldots,(x_N^s,a_N^s))_{s\le t}\in H^{t+1}$$

is *terminal* if $t=L$. Let Z^t be the set of terminal histories at t, and $Z\equiv\bigcup_{t=1}^{T+1}Z^t$ be the set of all terminal histories.

A *strategy* of bidder i is a function $\sigma_i: H\backslash Z\to X_i\times\{0,1\}$ satisfying the bidding constraint: for any $h^t\in H^t$, $x_i^t\le x_i^{t-1}$ where $\sigma_i(h^t)=(x_i^t\ a_i^t)$ and x_i^{t-1} is i's bid at $t-1$ in h^t. For each $i\in N$, the set of strategies is defined by $\sum_i$.

Definition 1. *Bidder i's* sincere-demand *at price* $p\in\mathbb{Z}_+$ *is defined by*

$$Q_i(p)=\min\left\{\arg\max_{x_i\in X_i}(U_i(x_i)-px_i)\right\}.$$

Definition 2. *Bidder i's* sincere-bidding *is the strategy such that for any* $t\ge 1$ *and* $h^t\in H^t\backslash Z^t$,

$$\sigma_i^*(h^t)=(\min\left\{x_i^{t-1},\max\left\{Q_i(p^t),C_i^{t-1}\right\}\right\},1_{(x_i^{t-1})\le Q_i(p^{t-1})}),$$

and $\sigma_i^*(h^0)=(Q_i(p^0),1)$. *Note that* $1_{(x_i^{t-1})\le Q_i(p^{t-1})}$ *is the indicator function, that is,*

$$1_{(x_i^{t-1})\le Q_i(p^{t-1})}=\begin{cases}1 & \text{if } x_i^{t-1}\le Q_i(p^{t-1})\\ 0 & \text{otherwise.}\end{cases}$$

For any history, sincere-bidding bids the sincere-demand as long as it satisfies the bidding constraint, and reports 0 as a signal if and only if the bidder overbids just before the history.

A *strategy profile* $(\sigma_i)_{i\in N}$ is an *n*-tuple of strategies. *Sincere-bidding by all bidders* $(\sigma_i^*)_{i\in N}$ is the *n*-tuple of sincere bidding.

2-4. Subgames

In this paper, we assume that at each round, each bidder can observe all the previous bids of the others.[3] Therefore, for each non-terminal history $h\in H\backslash Z$, we can define the subgame that follows the history h.[4]

For each h, let

$$H|_h=\{h'\in H: h'=(h,h'') \text{ for some sequence } h''\}.$$

be the set of all histories in the subgame that follows h, and

[3] This assumption is often called the *full bid information.*

[4] See, for example, Osborne and Rubinstein (1994) for details of subgames and induced strategies.

$$Z|_h = Z \bigcap H|_h$$

be the set of all terminal histories in the subgame that follows h. For each h and each $\sigma_i \in \sum_i$, the *induced strategy* is denoted by $\sigma_i|_h$, and let $\sum_i|_h$ be the set of induced strategies in the subgame that follows h.

2-5. Equilibrium concept

In extensive form games, a famous equilibrium notion is *subgame perfect equilibrium,* which is introduced by Selten (1975). On the other hand, we sometimes investigate *ex-post equilibrium* in auction games.[5] Ausubel (2004) combines these two concepts and introduces the notion of *ex post perfect equilibrium.*

Definition 3. *A strategy profile $(\sigma_i)_{i\in N}$ is an ex-post perfect equilibrium if for any history $h \in H \setminus Z$, the induced strategy profile $(\sigma_i|_h)_{i\in N}$ is an ex post equilibrium in the subgame that follows h.*

3. Result

Theorem 1. *In the Ausubel auction with the receive option of over-assignments, sincere-bidding by all bidders is always an ex-post perfect equilibrium which obtains the Vickrey outcome.*

Proof of Theorem 1. Consider any time $t \in \{0, 1, ..., T\}$, any non-terminal history

$$h^t = ((x_1^s, a_1^s), (x_2^s, a_2^s)..., (x_N^s, a_N^s))_{s \le t-1} \in H^t \setminus Z^t$$

and any marginal utilities $(u_j)_{j\in N}$. For each $j \in N$, let σ_j^* be sincere-bidding which is corresponding to u_j, and $\sigma_j^*|_{h^t}$ be induced sincere-bidding in the subgame that follows h^t.

If $x_i^{t-1} \le Q_i(p^{t-1})$, then we can show that the bidder i has no incentive to deviate by the similar way to Lemma 2 of Okamoto (2018). Then, we suppose that

$$x_i^{t-1} > Q_i(p^{t-1}) \tag{1}$$

that is, bidder i overbids at t in h^t.

Take any $i \in N$ and any strategy $\sigma_i \in \sum_i|_{h_t}$ of subgame that follows h^t. Let

$$z^{L+1} = \Big((x_1^s, a_1^s), (x_2^s, a_2^s), ..., (x_n^s, a_n^s) \Big)_{s \le L}$$

be the terminal history which is reached by $(\sigma_j^*|_{h^t})_{j\in N}$, and

[5] For example, See Crémer and Richard (1985).

$$w^{L'+1} = \Big((\hat{x}_1^s, \hat{a}_1^s), (\hat{x}_2^s, \hat{a}_2^s), ..., (\hat{x}_n^s, \hat{a}_n^s) \Big)_{s \leq L'}$$

be the terminal history which is reached by $(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i})$. Denote $\left\{(C_j^t)_{j \in N}\right\}_{t=0}^{L}$ the cumulative clinches of z^{L+1}, and $\left\{(\hat{C}_j^t)_{j \in N}\right\}_{t=0}^{L'}$ the cumulative clinches of $w^{L'+1}$.

We shall consider the three cases: (Case 1) $L > t$, (Case 2) $L' > L = t$, and (Case 3) $L' = L = t$. Note that the proof technique of Cases 1 and 2 is the same as the proof of Proposition 1 of Okamoto (2018).

Case 1. $L > t$.

Claim 1-1. $x_i^{L-1} \neq C_i^{L-2}$ or $C_i^{L-1} = 0$.

Suppose that $x_i^{L-1} = C_i^{L-1}$and $C_i^{L-1} > 0$. By $C_i^{L-1} \neq 0$ and the definition of cumulative clinches, $C_i^{L-2} = M - \sum_{j \neq i} x_i^{L-2}$. Therefore, $x_i^{L-1} = M - \sum_{j \neq i} x_i^{L-2}$. By bidding constraint for each $j \in N$, $x_j^{L-1} \leq x_j^{L-2}$. Hence, $\sum_{i \in N} x_i^t \leq M$.This means that the auction ends at $L-1$. This is a contradiction.

Claim 1-2. $x_i^{L-1} \leq Q_i(p^{L-1})$.

By the definition of sincere-bidding,

$$x_i^{L-1} = \min\left\{x_i^{L-2}, \max\left\{C_i^{L-2}, Q_i(p^{L-1})\right\}\right\}.$$

Then, by Claim 1-1,

$$x_i^{L-1} = \min\left\{x_i^{L-2}, Q_i(p^{L-1})\right\}.$$

Hence, $x_i^{L-1} \leq Q_i(p^{L-1})$.

Claim 1-2 is the same argument as step 1 of Lemma 2 of Okamoto (2018). Therefore, by using Claim 1-2, we can show that bidder i has no incentive to deviate similarly to Lemma 2 of Okamoto (2018).

Case 2. $L' > L = t$.

For each $j \in N$ and each $s \leq t-1$, $x_j^s = \hat{x}_j^s$, because there are parts of h^t.
Hence, for each $s \leq t-1 = L-1$, $C_i^s = \hat{C}_i^s$. Then, we shall calculate C_i and $\left\{\hat{C}_i^s\right\}_{s=L}^{L'}$.

Claim 2-1. $Q_i(p^L) \leq C_i^L$.

By the definition of sincere-bidding,

$$x_i^L = \min\left\{x_i^{L-1}, \max\left\{C_i^{L-1}, Q_i(p^L)\right\}\right\}.$$

By the bidding constraint and the definition of the sincere-demand,

$$x_i^{L-1} \geq C_i^{L-1}$$

$$x_i^{L-1} > Q_i(p^{L-1}) \geq Q_i(p^L).$$

Then,

$$x_i^L = \max\left\{C_i^{L-1}, Q_i(p^L)\right\}.$$

Therefore, $Q_i(p^L) \leq x_i^L$. By the definition of assignments, $x_i^L \leq C_i^L \leq x_i^{L-1}$.
Hence, $Q_i(p^L) \leq C_i^L$.

Claim 2-2. $C_i^L \leq \hat{C}_i^L$.

Since $t=L$, for each $j \neq i$, $x_j^L = \sigma_j^*|_{h^t}(h^t)$ and $\hat{x}_j^L = \sigma_j^*|_{h^t}(h^t)$. Then, for each $j \neq i$, $x_j^L = \hat{x}_j^L$. Since $L' > L$, the auction does not end at L in the history $w^{L'+1}$.
Then,

$$\hat{C}_i^L = M - \sum_{j \neq i} \hat{x}_j^L = M - \sum_{j \neq i} x_j^L.$$

Since the auction ends at L in the history z^{L+1},

$$C_i^L \leq x_i^L \leq M - \sum_{j \neq i} x_i^L.$$

Therefore, $C_i^L \leq \hat{C}_i^L$.

By Claim 2-1 and Claim 2-2, $Q_i(p^L) \leq C_i^L \leq \hat{C}_i^L$. Furthermore, for all $s \geq L+1$, $Q_i(p^s) \leq \hat{C}_i^s$, because $Q_i(p^s) \leq Q_i(p^L)$ and $\hat{C}_i^L \leq \hat{C}_i^s$. Therefore, i's utility in z^{L+1} is not less than that in $w^{L'+1}$.

Case 3. $L' = L = t$.

Similarly to Case 2, we show that for each $s \leq t-1 = L-1$, $C_i^s = \hat{C}_i^s$. Then, we shall calculate C_i^L and $\hat{C}_i^L$.

Claim 3-1. $C_i^L = x_i^L$.

By the definition of sincere-bidding and $x_i^{L-1} > Q_i^{L-1}$,

$$a_i^L = 0.$$

Therefore, by the auction rule, i's assignment in z^{L+1} is the same as i's last bid x_i^L, and so $C_i^L = x_i^L$.

If $x_i^L = Q_i(p^L)$, then obviously i's utility in z^{L+1} is not less than that in $w^{L'+1}$. Then, we suppose $x_i^L = C_i^{L-1}$. Hence, $C_i^L = C_i^{L-1}$. By the definition of sincere-bidding, $C_i^{L-1} \geq Q_i(p^L)$. Thus, $C_i^L \geq Q_i(p^L)$. By the definition cumulative clinches, $\hat{C}_i^L \geq \hat{C}_i^{L-1}$. Therefore, $\hat{C}_i^L \leq C_i^L \leq Q_i^L$, and so i's utility in z^{L+1} is not less than that in $w^{L'+1}$.

On the equilibrium path, all bidders never overbid at each round by the definition of sincere-bidding, and so all signals are 1. If all signals are 1, the outcome of the Ausubel auction with the receive option of over-assignments is the same as the outcome of original Ausubel auction with proportional rationing. Therefore, sincere-bidding equilibrium obtains the Vickrey outcome.

4. Conclusion

We have proposed a new auction, the Ausubel auction with the receive option of over-assignments. We have shown that in this auction, sincere-bidding equilibrium is an ex-post perfect equilibrium and yields the Vickrey outcome. Here, we compare our new rule with the Ausubel auction with sequential rationing. In the Ausubel auction with sequential rationing, the bidders have to be prioritized by some way. Therefore, as a priority changes, does an allocation in that auction.[6]

On the other hand, our new auction does not need a priority on bidders. In fact, if the bids of two bidders are the same, then the assignments of two bidders are the same. In this sense, our auction treats all bidders equally. It is sure that there may be some unsold objects in our auction. However, on a sincere-bidding equilibrium path, our auction are able to sell all objects, and it obtains an efficient outcome because it is the Vickrey outcome.

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[6] Note that on an sincere-bidding equilibrium path, the utilities of all bidders never change even if a priority changes.