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Parametric estimation of first-price auctions using approximate Bayesian computation

Yohsuke Hirose*

Abstract

In the structural estimation of first-price auctions, the likelihood of valuations is unavailable because bidders' valuations are unobserved. In order to overcome this problem, we use the approximate Bayesian computation (ABC). In this paper, we consider the estimation problem of the first-price auction model by imposing parametric specifications. Combining the ABC and Markov Chain Monte Carlo (MCMC) simulation method, we estimate the parameters of the density of valuations. We also conduct Monte Carlo experiments to evaluate our estimation method.

1. Introduction

Beginning with Paarsch (1992), numerous studies have proposed various methods for estimating the distribution of valuations. One of the difficulties in estimating structural parameters in the first-price auction model is that it is often impossible to calculate the likelihood, because it is difficult to evaluate the inverse of the equilibrium bidding function. In this paper, instead of computing the likelihood, we use the approximate Bayesian computation (ABC) to estimate the structural parameters of the auction model. Generally, Bayesian inference requires the evaluation of likelihood. However, because bidders' valuations are not observable, the likelihood of valuations is not available in most auction models. The ABC is a method that approximates the posterior distribution when the likelihood is not available.[1] In particular, imposing a parametric specification on the auction model, we use the ABC-MCMC method proposed by Marjoram et al. (2003) in this paper.

Bayesian estimation has been used in a variety of models in the literature of structural estima-

* Faculty of Economics, Meiji Gakuin University, 1-2-37 Shirokane-dai Minato-ku, Tokyo, 108-8636, Japan. E-mail: yhirose@eco.meijigakuin.ac.jp

1 Sisson and Fan (2018) is an excellent survey of ABC.

tion of first-price auctions. Li and Zheng (2009) and Li and Zheng (2012) applied the Bayesian approach to auction models with entry. Kumbhakar et al. (2012) assumed that there is an error between the equilibrium bid and the actual bid and used Bayesian estimation for the error distribution. Kim (2015) considered the affiliated private values model using a series representation to specify the density of valuation. Further, Aryal et al. (2018) applied Bayesian approach to estimate the effect of ambiguity aversion in first price auctions.

The remainder of the paper is organized in the following manner. In Section 2, we review the ABC method briefly. In Section 3, we present a standard first-price auction model. We discuss the relationship between bidders' valuations and observed equilibrium bids in the first-price auction model. In Section 4, we explain the estimation strategy. As explained, we apply the ABC-MCMC method to estimate the density of valuations. In Section 5, we conduct simulation experiments to demonstrate our estimation method discussed in Section 4. In Section 6, we present the conclusion to the article.

2. Approximate Bayesian Computation

In this section, we briefly review the ABC-MCMC method. Generally, by the Bayes rule, the posterior density of parameter θ, $\pi(\theta|y)$ is given by

$$\begin{aligned}\pi(\theta|y) &= \frac{p(y|\theta)\pi(\theta)}{p(y)} \\ &\propto p(y|\theta)\pi(\theta),\end{aligned}$$

where y is observations, $p(y|\theta)$ is the likelihood, $\pi(\theta)$ is the prior density of θ, and $p(y)$ is the marginal likelihood.

In ABC, we usually assume that we can generate random draws, z, from the likelihood $p(\cdot|\theta)$. Then, the approximated density by ABC is described as

$$\begin{aligned}\pi_{\mathrm{ABC}}(\theta|y) &\propto \int 1\{\|y-z\| \leq \epsilon\} p(z|\theta)\pi(\theta)\,dz \\ &= \int \pi_{\mathrm{ABC}}(\theta, z|y)\,dz,\end{aligned}$$

where ϵ is a tolerance level and $\pi_{\mathrm{ABC}}(\theta, z|y) \propto 1\{\|y-z\| \leq \epsilon\} p(z|\theta)\pi(\theta)$ is the joint approximate posterior of (θ, z). Note that if $\epsilon = 0$, $\pi_{\mathrm{ABC}}(\theta|y) = \pi(\theta|y)$ holds. In the Metropolis-Hastings (MH) algorithm, we require the proposal distribution $q((\theta^*, z^*) | (\theta, z))$. In ABC-MCMC, the proposal distribution $q((\theta^*, z^*) | (\theta, z))$ assumes the following form:

$$q((\theta^*, z^*) | (\theta, z)) = q(\theta^*|\theta)p(z^*|\theta^*),$$

where θ^* is a candidate parameter and z^* is a random draw from $p(z^*|\theta^*)$. Then, the acceptance rate in the MH step is evaluated as

$$\alpha(\theta, \theta^*) \equiv \min\left[1, \frac{\pi_{\mathrm{ABC}}(\theta^*, z^* | y) q((\theta, z) \mid (\theta^*, z^*))}{\pi_{\mathrm{ABC}}(\theta, z | y) q((\theta^*, z^*) \mid (\theta, z))}\right]$$

$$= \min\left[1, \frac{q(\theta | \theta^*) \pi(\theta^*)}{q(\theta^* | \theta) \pi(\theta)} \frac{1\{\|y - z^*\| \leq \epsilon\}}{1\{\|y - z\| \leq \epsilon\}}\right]. \tag{1}$$

Note that in the RHS of Eq. (1), the likelihood $p(\cdot \mid \theta)$ does not appear. Therefore, we can run the MCMC simulations without evaluating the likelihood. We regard θ^* as a random draw from posterior distribution with probability $\alpha(\theta, \theta^*)$ and reject θ^* with probability $1 - \alpha(\theta, \theta^*)$.

3. First-price auction

We consider a first-price auction within the independent private values (IPV) paradigm. A seller offers a single good for sale at a reserve price r_0. Reserve prices are observable for all bidders. There are N potential bidders. In this paper, N is observable for all bidders. Let V_i denote bidder i's valuation for the item, which are private values, and independently and identically distributed among $i \in 1, \ldots, N$. Let $F(\cdot)$ denote the distribution function of bidders' valuations whose support is $[\underline{v}, \bar{v}]$, and $f(\cdot)$ denotes its density.

Bidder i submits a bid, $b_i > 0$. The bidder who submits the highest bid amount wins the item and pays her bid. On the other hand, bidders who do not win pay nothing. Let $u(b_i | v_i)$ be bidder i's payoff function when she bids b_i and her willingness-to-pay is v_i. Bidders are assumed to be risk-neutral. In other words, bidder i's payoff $u(b_i | v_i)$ is described in the following manner:

$$u(b_i | v_i) = \begin{cases} v_i - b_i & \text{if bidder } i \text{ wins, and} \\ 0 & \text{otherwise.} \end{cases}$$

Then, as is well known, the symmetric equilibrium bidding function $\beta(\cdot)$ is given by

$$\beta(v) = v - \frac{1}{[F(v)]^{N-1}} \int_{\max\{\underline{v}, r_0\}}^{v} [F(u)]^{N-1} du \quad \text{for } v > \max\{\underline{v}, r_0\}. \tag{2}$$

Most of studies on the structural estimation of first- price, sealed-bid auctions assume that Eq. (2) holds between the valuation v and the observed bid b. In other words, Eq. (2) can be considered part of the data generation process.

4. Estimation procedure

Let L be the number of observed auctions, indexed by $l=1, \ldots, L$. In each auction $l \in \{1, \ldots, L\}$, there are N_l potential bidders. In this paper, N_l is observable for econometricians. In addition, we observe the reserve price of auction l, r_l, and the bid amount of bidder i in auction l, b_{li}. In this paper, we impose a parametric specification on the distribution of valuations. In other words, we assume that the distribution function of valuation $F(\cdot \mid \theta)$ is known up to θ, and we estimate unknown parameter θ from observations.

Letting v_{li} be bidder i's valuation in auction l, $b_{li} = \beta(v_{li})$ holds because b_{li} is the equilibrium bid. Thus, evaluating $v = \beta^{-1}(b)$, the parameter vector θ can be estimated from observations. However, because the integration of Eq. (2) cannot be calculated analytically, the inverse function $\beta^{-1}(\cdot)$ will also be difficult to compute. In this paper, we propose a likelihood-free approach to estimate the parameter vector θ from observations. One of the likelihood-free approaches is the ABC. Using the ABC, we compute an approximate posterior distribution without computing the likelihood. We first consider the case that we can observe all the bids. In other words, the observation is $b \equiv (b_{11}, \ldots, b_{1N_1}, b_{2,1}, \ldots, b_{L,1}, b_{L,N_L})$. Then, the likelihood of observed bids b, $p(b \mid \theta)$ is given by

$$p(b|\theta) = \prod_{l=1}^{L} \prod_{i=1}^{N_l} g(b_{li}|\theta) , \tag{3}$$

where $g(b_{li} \mid \theta)$ denotes the density function of bidder i's bid in auction l. From Eq (2) and by the inverse function theorem, $g(b|\theta)$ is described as

$$g(b|\theta) = f(v|\theta) \cdot \frac{1}{d\beta(v;\theta)/dv} .$$

However, as discussed above, because evaluating $v=\beta^{-1}(b)$ is difficult, the likelihood (3) is also not computable.

Now, we describe the procedure in the t-th iteration of ABC-MCMC. First, we draw a candidate parameter, θ^*, from the proposal distribution $q(\theta^*|\theta^{(t-1)})$, where $\theta^{(t-1)}$ is the parameter obtained in iteration $t-1$. Next, for each $i \in \{1, \ldots, N_l\}$ and each $l \in \{1, \ldots, L\}$, we generate a candidate valuation v^*_{li} from $f(\cdot|\theta^*)$. Then, from Eq. (2) and v^*_{li}, we can calculate the candidate bid of bidder i in auction l, b^*_{li}. In the ABC step, intuitively, we accept b^*_{li} and θ^* if the candidate bid b^*_{li} and observed bid b_{li} are sufficiently close. More precisely, using observed bid b_{li}, the acceptance rate in the MH step can be calculated by

$$\alpha(\theta^{(t-1)}, \theta^*) \equiv \min\left[1, \frac{q(\theta^{(t-1)}|\theta_j^*)\pi(\theta^*)}{q(\theta^*|\theta^{(t-1)})\pi(\theta^{(t-1)})} \cdot \frac{1\{\sum_{l=1}^{L}\sum_{i=1}^{N_l}|b_{li}-b_{li}^*| \leq \epsilon\}}{1\{\sum_{l=1}^{L}\sum_{i=1}^{N_l}|b_{li}-b_{li}^{(t-1)}| \leq \epsilon\}}\right], \tag{4}$$

where $\pi(\cdot)$ is the prior distribution, $b_{li}^{(t-1)}$ is the (accepted) candidate bid obtained in the $(t-1)$-th step (i.e., the candidate bid corresponding to $\theta^{(t-1)}$) and ϵ is a positive small value. The candidate parameter θ^* is accepted with probability $\alpha(\theta^{(t-1)}, \theta^*)$ and rejected with probability $1-\alpha(\theta^{(t-1)}, \theta^*)$.

To summarize, the procedure can be implemented in the following manner:

1. Set initial parameters: $\theta^{(0)}$.
2. For $t = 1, \ldots, T$, repeat the following steps:
   (i) Generate a candidate parameter θ^* from $q(\theta^*|\theta^{(t-1)})$.
   (ii) Generate a candidate valuation v_{li}^* from $f(v|\theta^*)$ for $l \in \{1, \ldots, L\}$ and $i \in \{1, \ldots, N_l\}$.
   (iii) Compute a candidate bid b_{li}^* by Eq. (2) for $l \in \{1, \ldots, L\}$ and $i \in \{1, \ldots, N_l\}$.
   (iv) Calculate the acceptance rate $\alpha(\theta^{(t-1)}, \theta^*)$ as in Eq. (4).
   (v) Set

$$\theta^{(t)} = \begin{cases} \theta^* & \text{with probability } \alpha(\theta^{(t-1)}, \theta^*) \\ \theta^{(t-1)} & \text{with probability } 1-\alpha(\theta^{(t-1)}, \theta^*) \ . \end{cases}$$

Let $T^{\text{burn-in}}$ denote the burn-in-period. We regard $\theta^{(T^{\text{burn-in}}+1)}, \ldots, \theta^{(T)}$ as random draws from posterior distribution.

4-1. Extension: When only the winning bids are observable

Often, there are auctions where only the winning bid is observable. In this subsection, we consider the case where only the winning bids can be observed. Let w_l be the winning bid of auction $l \in \{1, \ldots, L\}$. Then, because the winning bid of auction l is the largest order statistics of all the bids $b_{l1}, \ldots, b_{lN_l}$, the likelihood of observed winning bids $w \equiv (w_1, \ldots, w_L)$ is described in the following manner

$$p(w|\theta) = \prod_{l=1}^{L} g(w_l|\theta)[G(w_l|\theta)]^{N_l-1},$$

where $G(\cdot|\theta)$ denotes the distribution function of bidder i's bid in auction l. Similar to likelihood (3), this likelihood is also not computable because $\beta^{-1}(\cdot)$ cannot be calculated.

The ABC-MCMC method is still applicable in this case. We describe the procedure in the t-th iteration of ABC-MCMC. First, we draw a candidate parameter vector, θ^*, from the proposal distribution $q(\theta^*|\theta^{(t-1)})$. Next, for each $i \in \{1, \ldots, N_l\}$ and each $l \in \{1, \ldots, L\}$, we generate a candidate valuation v_{li}^* from $f(\cdot|\theta^*)$. Then, from Eq. (2) and v_{li}^*, we compute the candidate bid of bidder i in auction l, b_{li}^*. In addition, the winning bid of auction l can be computed as $w_l = \max\{b_{l1}, \ldots, b_{lN_l}\}$ for each $l \in \{1, \ldots, L\}$. Then, using the observed winning bid, w_l, the acceptance rate in the MH step can be calculated by

$$\alpha(\theta^{(t-1)}, \theta^*) \equiv \min\left[1, \frac{q(\theta^{(t-1)}|\theta_j^*)\pi(\theta^*)}{q(\theta^*|\theta^{(t-1)})\pi(\theta^{(t-1)})} \cdot \frac{1\{\sum_{l=1}^{L}|w_l - w_l^*| \le \epsilon\}}{1\{\sum_{l=1}^{L}|w_l - w_l^{(t-1)}| \le \epsilon\}}\right],$$

where $w_l^{(t-1)}$ is the (accepted) candidate winning bid obtained in the $(t-1)$-th step. The candidate parameter vector θ^* is accepted with probability $\alpha(\theta^{(t-1)}, \theta^*)$ and rejected with probability $1-\alpha(\theta^{(t-1)}, \theta^*)$.

5. Monte Carlo experiments

In order to demonstrate our estimation strategy, we conducted two Monte Carlo experiments. Through this section, the number of potential bidders is N=5 for all $l \in \{1, \ldots, L\}$. In the experiments, we do not set the reserve price (i.e., $r_0 = \underline{v}$). The computational results were obtained by using the Julia programming language.

5-1. Case 1: When all bids are observable

In this subsection, we consider the auctions where all bids are observable. We set the number of auctions, L=20. Therefore, in our experiments, we observe $J \equiv N \times L$=100 bids. A symmetric environment enables us to index observed bids by $b_1, \ldots, b_j, \ldots, b_{100}$. First, we generated the valuations $v_1, \ldots, v_{100}$ from the beta distribution with parameter $(\alpha, \beta) = (2, 3)$ —that is,

$$V_j \sim i.i.d.\ \mathrm{Beta}(2, 3).$$

Next, we computed the bids $b_1, \ldots, b_{100}$ from Eq. (2).

The observations are the number of bidders N=5 and the bidders' bids $b \equiv (b_1, \ldots, b_{100})$. From the observations, we estimated the true parameter $(\alpha, \beta) = (2, 3)$ by the estimation procedure described in Section 4. We set the prior distributions of α and β in the following manner:

$$\alpha \sim \mathrm{Gamma}(0.1, 10), \beta \sim \mathrm{Gamma}(0.1, 10).$$

We set the tolerance level for the ABC step as $\epsilon = 0.5$. The number of iterations of ABC-MCMC is 55000. The burn-in-period is 5000. Thus, we obtain 50000 random draws from the ABC-MCMC simulation method.

Table 1 presents the estimation results for (α, β). The column entitled "True value" presents the true values. The posterior means, the posterior standard deviations, and the 95% credible intervals are presented in the columns entitled "Mean," "St. dev.," and "95% interval," respectively. Our estimator included true values in 95% credible intervals. Further, all p-values of the convergence diagnostics (CD) were over 0.05. The values of the inefficiency factors (IF) did not exceed 300, which is sufficiently low. In summary, it can be concluded that our estimation method performed well.

Table 1: Estimation results (Case 1)

	True value	Mean	St. dev.	95% interval	CD	IF
α	2.00	2.10	0.722	(1.03, 3.70)	0.195	273
β	3.00	3.07	1.12	(1.44, 5.56)	0.159	284

Figures 1-2 represent the sample paths of the posterior draws. Figure 1 illustrates the sample path of α and Figure 2 indicates that of β. It is evident from these figures that the random draws converge to the posterior distributions. Figure 3 indicates the estimated posterior density of α and Figure 4 indicates that of β.

Figure 1: Sample path of α (Case 1)

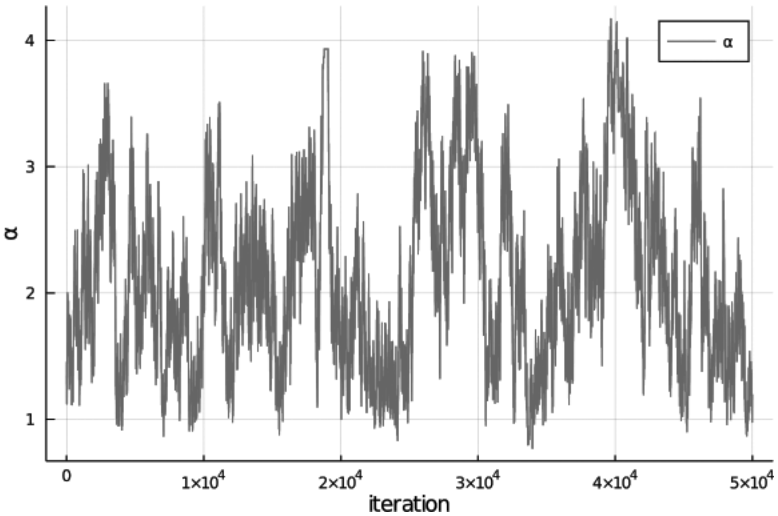


Figure 2: Sample path of β (Case 1)

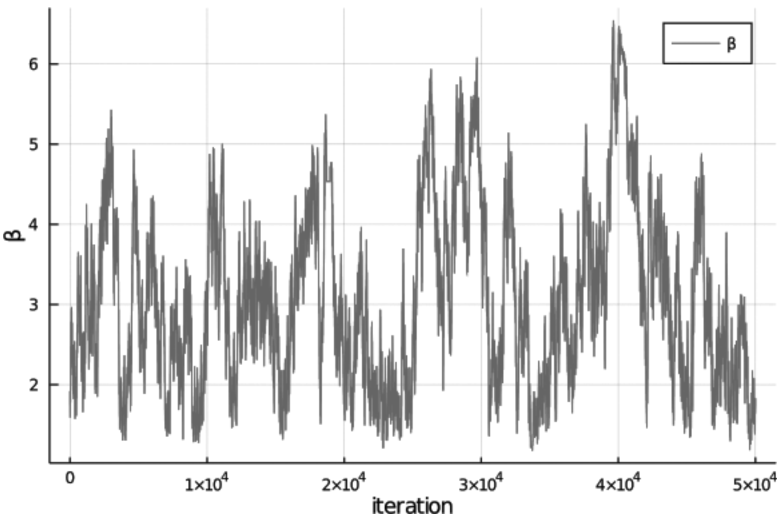


Figure 3: Posterior density of α （Case 1）

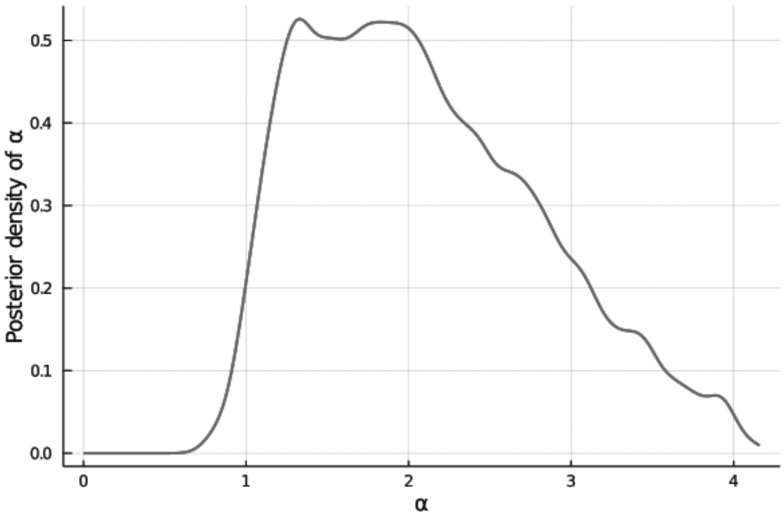
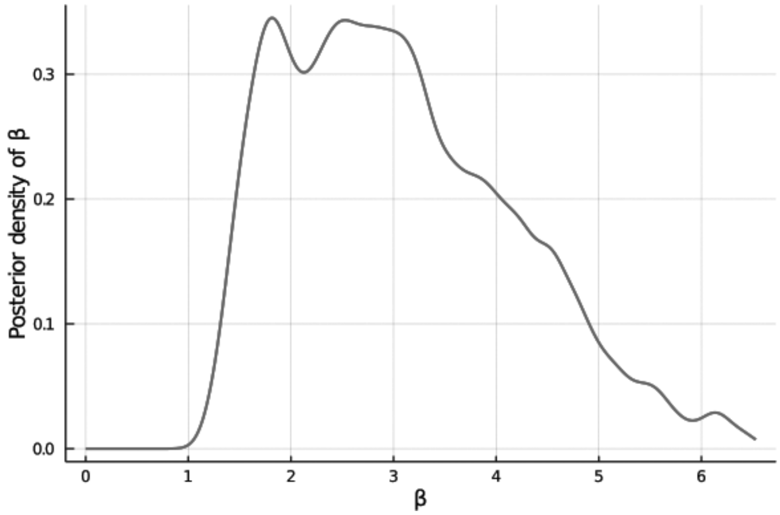


Figure 4: Posterior density of β （Case 1）



5-2. Case 2: When only winning bids are observable

In this Monte Carlo experiment, we consider the auctions where only winning bids are observable. The number of auctions is L=100. Thus, in this experiment, we observe 100 winning bids. First, we generated bidder i's valuation in auction l, v_{li} for all $i \in \{1, \ldots, 5\}$ and for all $l \in \{1, \ldots, 100\}$ from the beta distribution with parameter $(\alpha, \beta) = (2, 3)$ —that is,

$$V_{li} \sim i.i.d. \,\mathrm{Beta}(2, 3).$$

Next, we computed bidder i's bid in auction l, b_{li}, from Eq. (2). Finally, we computed the winning bid in auction l as $w_l = \max\{b_{l1}, \ldots, b_{l5}\}$.

The observations are the number of bidders $N=5$ and the winning bids $w \equiv (w_1, \ldots, w_{100})$. From the observations, we estimated the true parameter $(\alpha, \beta)=(2, 3)$ by the estimation method described in Section 4. We set the prior distributions of α and β in the following manner:

$$\alpha \sim \text{Gamma}(0.1, 10), \beta \sim \text{Gamma}(0.1, 10).$$

The tolerance level for ABC step is determined by $\epsilon = 0.15$. We iterated the ABC-MCMC procedure 75000 times. The burn-in-period is 5000. Thus, we obtain 70000 random draws from the ABC-MCMC simulation method.

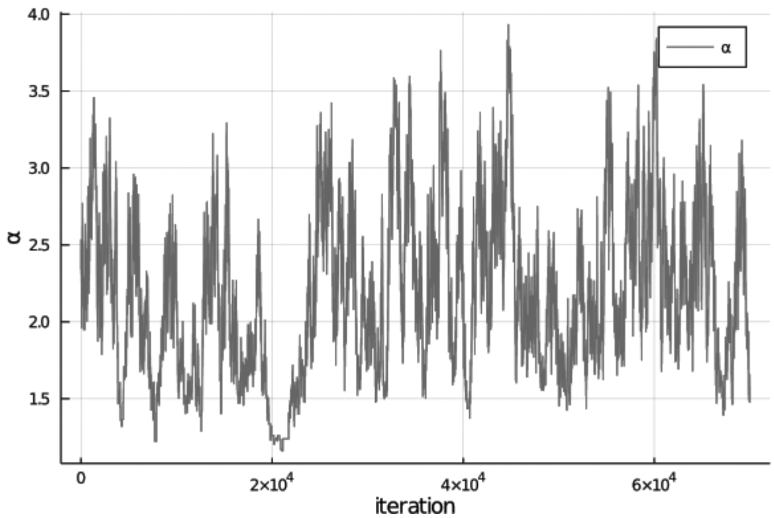
Table 2 presents the estimation results for (α, β). Our estimator included true values in 95% credible intervals. Further, all p-values of the convergence diagnostics (CD) were over 0.05. The values of inefficiency factors (IF) were 320, which is sufficiently low. Similar to case 1, we can evaluate that our estimation method worked well.

Table 2: Estimation results (Case 2)

	True value	Mean	St. dev.	95% interval	CD	IF
α	2.00	2.18	0.537	(1.26, 3.35)	0.870	323
β	3.00	3.22	0.644	(2.14, 4.64)	0.831	323

Figure 5 presents the sample path of posterior random draws of α. Similarly, Figure 6 presents the sample path of posterior random draws of β. It is evident from these figures that the random draws converge to the posterior distributions. Figure 7 indicates the estimated posterior density of α and Figure 8 indicates that of β.

Figure 5: Sample path of α (Case 2)



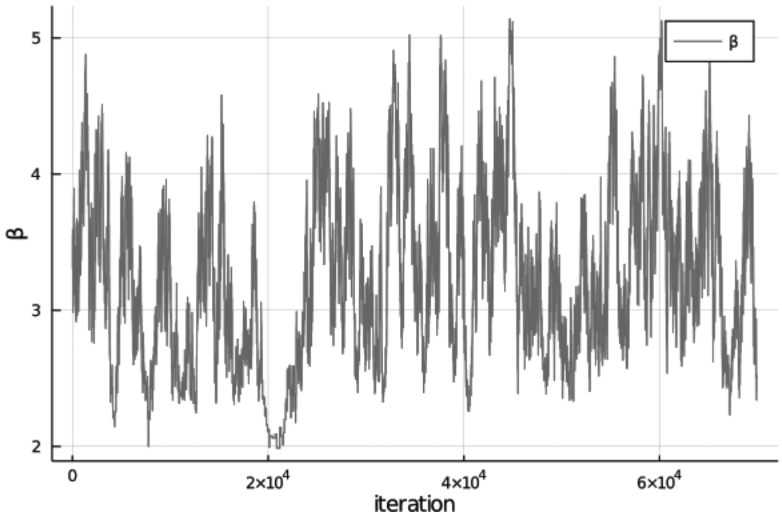


Figure 6: Sample path of β (Case 2)

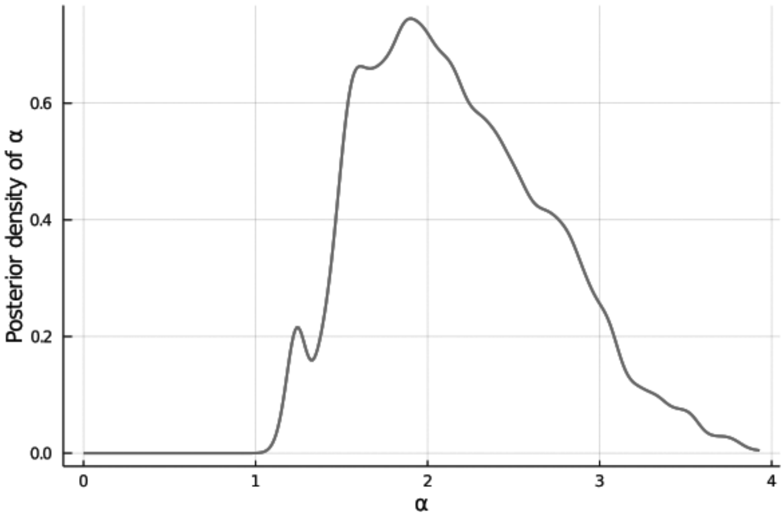


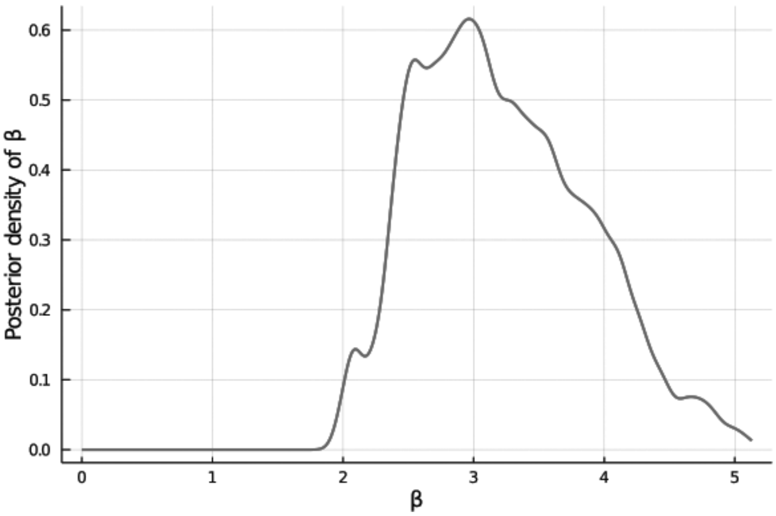
Figure 7: Posterior density of α (Case 2)

6. Conclusion

In this paper, we propose a Bayesian approach to estimate the parameters of the density of valuations. In most cases of the structural estimation of first-price auctions, the likelihood is not available because the valuations are unobserved. In order to overcome this problem, we applied the ABC method. The ABC method is used to approximate the posterior distribution when the likelihood is not available. We estimated the structural parameters by combining the ABC and MCMC methods.

We also conducted Monte Carlo experiments to demonstrate our estimation method. Two experiments are conducted. In the first experiment, we generated valuations from the beta distribution

Figure 8: Posterior density of β (Case 2)



with true parameter vector $(\alpha, \beta) = (2, 3)$ and computed equilibrium bids. The observation is all the bids and the number of potential bidders. In the second experiment, we generated valuations from the beta distribution with parameters $\alpha=2$ and $\beta=3$ and computed equilibrium bids. The observation is only the winning bids and the number of potential bidders. In both the experiments, from the observations, we estimated the parameters of the density of valuations, (α, β). The results of our experiments reveal that our proposed estimation method can work well.

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