An Unbalanced Multi-sector Growth Model with Constant Returns: A Turnpike Approach

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09-03

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(May, 2009)

*The paper is prepared for the International Conference on " Globalization, interdependences and macroeconomic fluctuations", held in Paris, June 11-13, 2009. The paper is very preliminay. Please do not quote.

1 Introduction

Since the seminal papers by Romer (1986) and Lucas (1988), we have witnessed a strong revival of interest in Growth Theory under the name of Endogenous Growth Theory, and especially, neoclassical optimal growth models have been used as analytical benchmark models, which have been intensively studied in late 60's. However, these research models have a serious drawback. Since the models are based on the highly aggregated macro-production function, they cannot explain the important empirical evidence, as I will give a detailed discussion in the following section. Recent empirical studies at the industry level among countries provide a clear evidence that individual industry's per-capita capital stock and output grow at industry's own growth rate, which is closely related to industry's technical progress measured by the total factor productivity of the industry. For example, the per-capita capital stock and output of an agriculture sector grow at 5% per annum along its own steady state, on the other hand, those of a manufacturing sector grow at 10% per annum along its own steady state. Let us refer to this phenomenon as " unbalanced growth among industries". To tackle the problem, it has raised a strong theoretical demand for constructing a multi-sector growth model. In spite of strong needs for such a model, very little study of this type of model has been done so far.

On the other hand, the optimal growth model with heterogeneous capital goods

Scheinkman (1978) and McKenzie (1986)). Thus he still did not fully exploit the structure of the neoclassical optimal growth model, especially the dynamics of the path on the Neumann-McKenzie facet to obtain the Turnpike property.

The paper is undertaken to fill the gap between the results derived by the theoretical researches explained above and the empirical evidence provided by the recent empirical studies at the industry level among countries by way of applying the theoretical method developed in Turnpike Theory. I will first set up a multi-sector optimal growth model, where each sector exhibits the Harrod neutral technical progress with a sector specific rate. The presented model will be regarded as a multi-sector optimal growth version of the Solow model with the Harrod neutral technical progress. Secondly, I will rewrite the original model into a per-capita efficiency unit model. Then as the third step, I will transform the efficiency unit model into a reduced form model. Then the method developed in Turnpike Theory are ready to be applicable. I will first establish the Neighborhood Turnpike Theorem demonstrated in McKenzie (1983). The neighborhood Turnpike means that any optimal path will be trapped in a neighborhood of the corresponding optimal steady state path when discount factors are close enough to one and the neighborhood can be made as small as possible by choosing a discount factor arbitrarily close to one. Then, I will show the local stability by applying the logic used by Scheinkman (1976): there exists a stable manifold that stretches out over today's capital stock plane. To demonstrate both theorems, easily accessed on the Web: *the EU-Klems Growth and Productivity Database*[1], which covers 28 countries with 71 industries from 1970 to 2005. It contains the GDPs and the total factor productivity (TFPs) of industries. Growth accounting has been used to analyze economic growth in countries. One of the more interesting applications is to the industries. Let us assume the following production function of the ith industry in a country.

$$Y_i(t) = F^i(K_{1i}(t), K_{2i}(t), \cdots, K_{ni}(t), A_i(t)L_i(t)),$$

where Y_i : tth period capital goods output of the ith industry, K_{ji} : ith capital goods used in the jth industry in the tth period, A_t^i : tth period labor-argumented technical-progress, and $L_i(t)$: tth period labor input of the ith industry. If θ_j stands for the factor share of the jth input factor, then we may derive the following relation concerned with the ith industry;

$$\frac{\dot{A_i}}{A_i} = \frac{\frac{\dot{Y_i}}{Y_i} - \left(\sum_{j=1}^{n} \theta_{ji}\frac{\dot{K_{ji}}}{K_{ji}} + \theta_{0i}\frac{\dot{L_i}}{L_i}\right)}{\theta_{0i}}.$$

Based on this equation, we are able to caluculate TFPs of the 20 industries of a country[2]. Figures 1 show the relationship between the per-capita U.S. GDP average growth rate and the U.S. TFP average growth rate at the industry level from 1970 to

[1]URL http://www.euklems.net

[2]The U.S. 20 industries are followings:

1:TOTAL INDUSTRIES 2 :AGRICULTURE, HUNTING, FORESTRY AND FISHING

1) Each industrial sector has its own steady state with the sector specific growth rate.

2) The steady state level and its growth rate are highly related to its own TFP.

These facts cannot be explained by the new growth theory totally based on the macro production function. Thus we need to set up an industry based multi-sector growth model. On the other hand, the turnpike theory are established based on the multi-sector model. However it has a drawback, too. The turnpike theory means that each industrial sector with different initial stocks will eventually converge to its own optimal steady state with the common balanced growth rate. In other words, each industry's per capita stock will converges to a certain constant ratio. Thus the turnpike theory cannot explain the facts that each industry's per-capita stock grows at its own growth rate, which is determined by the sectoral TFP.

OECD (2003) also studied the productivity growth at the industry level in detail and reported the following results, which are consistent with our observations discussed above.

- A large contribution to overall productivity growth patterns comes from productivity changes within industries, rather than as a result of significant shifts of employment across industries.

- TFP depends on country/industry specific factors.

$$\sum_{i=0}^{n} L_i(t) = L(t), \tag{4}$$

$$\sum_{j=0}^{n} K_{ij}(t) = K_i(t), \tag{5}$$

where$i = 1, 2, ..., n, t = 0, 1, 2, ...,$ and the notation is as follows:

r = subjective rate of discount, r≥g,

$C(t) \in R_+$ = total consumption goods produced and consumed at t,

$Y_i(t) \in R_+$ = t^{th} period capital goods output of the i^{th} sector,

$K_i(t) \in R_+$ = t^{th} period capital stock of the i^{th} sector,

$K_i(0) \in R_+$ = initial capital stock of the i^{th} sector,

$F^j(\cdot) : R_+^{n+1} \longmapsto R_+$ = production function of the j^{th} sector,

$L_i(t)$ = t^{th} period labor input of the i^{th} sector,

$L(t)$ = t^{th} period total labor input,

$K_{ij}(t)$ = i^{th} capital goods used in the j^{th} sector in the t^{th} period,

δ_i = depreciation rate of the i^{th} capital goods, given as $0<\delta_i<1$,

$A_i(t)$ = t^{th} period labor-argumented technical-progress of the i^{th} sector.

Then,

$$\widetilde{y}_i(t) = f^i(\widetilde{k}_{1i}(t), \widetilde{k}_{2i}(t), \cdots, \widetilde{k}_{ni}(t), \ell_i(t)) \quad (i = 1, \cdots, n)$$

where $\widetilde{y}_i(t) = \frac{Y_i(t)}{A_i(t)L(t)}, \widetilde{k}_{1i}(t) = \frac{K_{1i}(t)}{A_i(t)L(t)}, \widetilde{k}_{2i}(t) = \frac{K_{2i}(t)}{A_i(t)L(t)}, \cdots, \widetilde{k}_{ni}(t) = \frac{K_{ni}(t)}{A_i(t)L(t)}$,and $\ell_i(t) = \frac{A_i(t)L_i(t)}{A_i(t)L(t)}$.

Applying the same transformation to the consumption sector, we have also

$$\widetilde{c}(t) = f^0(\widetilde{k}_{10}(t), \widetilde{k}_{20}(t), \cdots, \widetilde{k}_{n0}(t), \ell_0(t)).$$

Furthermore, we may also transform the t^{th} sector's accumulation equation as followas; dividing both sides by $A_t^i L(t)$,

$$\frac{Y_i(t)}{A_i(t)L(t)} + (1 - \delta_i)\frac{K_i(t)}{A_i(t)L(t)} - \frac{K_i(t+1)}{A_i(t)L(t)} = 0$$

Note the following relation:

$$\frac{K_i(t+1)}{A_i(t)L(t)} = \frac{(1+\alpha_i)(1+g)K_i(t+1)}{[(1+\alpha_i)A_i(t)][(1+g)L(t)]} = (1+\alpha_i)(1+g)\widetilde{k}_i(t+1).$$

Then we have finally,

$$\widetilde{y}_i(t) + (1 - \delta_i)\widetilde{k}_i(t) - (1+\alpha_i)(1+g)\widetilde{k}_i(t+1) = 0.$$

In a vector form expression,

$$\widetilde{\mathbf{y}} + (\mathbf{I} - \Delta)\widetilde{\mathbf{k}}(t) - (1+g)\mathbf{G}\widetilde{\mathbf{k}}(t+1) = 0$$

$$\widetilde{y}_i(t) = f^i(\widetilde{k}_{1i}(t), \widetilde{k}_{2i}(t), \cdots, \widetilde{k}_{ni}(t), \ell_i(t)) \ (i = 1, \cdots, n), \tag{7}$$

$$\widetilde{\mathbf{y}} + (\mathbf{I} - \Delta)\widetilde{\mathbf{k}}(t) - (1+g)\mathbf{G}\widetilde{\mathbf{k}}(t+1) = 0, \tag{8}$$

$$\sum_{i=0}^{n} \ell_i(t) = 1, \tag{9}$$

$$\sum_{i=0}^{n} \widetilde{k}_{ij}(t) = \widetilde{k}_j(t) \ (j = 1, \cdots, n). \tag{10}$$

We may add the following assumption and prove the basic property, Lemma 1;

Assumptin 3. $0 < \rho < 1$.

Lemma 1. Under Assumption 2, Eqs.(6)-(10) except Eq.(8) are summarized as the social production function $\widetilde{c}(t) = T(\widetilde{\mathbf{y}}(t), \widetilde{\mathbf{k}}(t))$ which is continuously differentiable on the interior $\mathbf{R}_+^{2n}$ and concave where $\widetilde{\mathbf{y}}(t) = (y_1(t), y_3(t), \cdots, y_n(t))$ and $\widetilde{\mathbf{k}}(t) = (k_1(t), k_2(t), \cdots, k_n(t))$.

Proof.

See Benhabib and Nishimura (1979). ■

where the partial derivative vectors mean that

$$\mathbf{V}_x(\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t+1)) = [\partial V(\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t+1))/\partial \widetilde{k}_1(t), \cdots, \partial V(\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t+1))/\partial \widetilde{k}_n(t)]^t,$$

$$\mathbf{V}_z(\widetilde{\mathbf{k}}(t-1), \widetilde{\mathbf{k}}(t)) = [\partial V(\widetilde{\mathbf{k}}(t-1), \widetilde{\mathbf{k}}(t))/\partial \widetilde{k}_1(t), \cdots, \partial V(\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t-1))/\partial \widetilde{k}_n(t)]^t$$

and $\mathbf{0}$ means an n dimensional zero column vector. " t " implies transposition of vectors. Note that under the differentiability assumptions, all the price vectors will satisfy the following relations:

$$\begin{aligned} q &= \partial\widetilde{c}/\partial\widetilde{c} = 1, \\ p_i &= -q\partial T(\widetilde{\mathbf{y}}, \widetilde{\mathbf{k}})/\partial \widetilde{k}_i \quad (i = 1, 2, \cdots, n), \\ w_i &= q\partial T(\widetilde{\mathbf{y}}, \widetilde{\mathbf{k}})/\partial \widetilde{k}_i \quad (i = 1, 2, \cdots, n). \\ w_0 &= q\widetilde{c} + \mathbf{p}\widetilde{\mathbf{y}} - \mathbf{w}\widetilde{\mathbf{k}} \end{aligned}$$

Using these relation, we may define the price vectors of capital goods as $(n \times 1)$ row vector $\mathbf{p} = (p_1, p_2, \cdots, p_n)$, the output of capital goods as $(n \times 1)$ vector $\widetilde{\mathbf{y}} = (\widetilde{y}_1, \widetilde{y}_2, \cdots, \widetilde{y}_n)^t$, the rental rate as $(1 \times n)$ row vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ and the capital stock as $(n \times 1)$vector $\widetilde{\mathbf{k}} = (\widetilde{k}_1, \widetilde{k}_2, \cdots, \widetilde{k}_n)^t$. w_0 is a wage rate. For simlicity we may assume that all the price vectors $(\mathbf{p}, \mathbf{w}, w_0)$ are expressed as the relative price vectors of the price of the consumption good q.

Definition. *An optimal steady state path* $\mathbf{k}^\rho$ (denoted by OSS henceforth) is an

$$[\mathbf{I} - (r\mathbf{I} + \Delta)\overline{\mathbf{A}}^{r}]^{-1} \geq \Theta$$

where Θ is a $n \times n$ zero matrix[3].

By the well known equivalence theorem of the Hawkins-Simon condition and Theorem 4 of Mckenzie (1960), Assumption 4 is equivalent to the property that the matrix $[\mathbf{I} - (r\mathbf{I} + \Delta)\overline{\mathbf{A}}^{r}]$ has a dominant diagonal that is positive; there exists $\mathbf{y} \geq \mathbf{0}$ such that $[\mathbf{I} - (r\mathbf{I} + \Delta)\overline{\mathbf{A}}^{r}]\mathbf{y} \geq \mathbf{0}$.

We need the following extra assumption.

Assumption 5. $1 > \alpha_0 > \max_{i=1,\dots,n} |\alpha_i|$

Remark 1 *The assumption means that the TFP growth rate of the consumption sector is the highest among those of sectors. Takahashi, Mashiyama and Sakagami (2004) reported that in the postwar Japanese economy, the consumption sector has exhibited a higer per-capita output growth rate than that of the capital goods sector in a two-sector model. If the TFP growth rate has a positive correlation with the per-capita sectoral GDP growth rate, this fact will partially support Assumption 5.*

[3]Let $\mathbf{A}$ and Θ be n-dimentional square matrix and n-dimensional zero matrix. Then $\mathbf{A} \gg \Theta$ if $a_{ij} > 0$ for all i, j, $\mathbf{A} > \Theta$ if $a_{ij} \geq 0$ for all i, j and $a_{ij} > 0$ for some i, j and $\mathbf{A} \geq \Theta$ if $a_{ij} \geq 0$ for all i, j.

$\widehat{\mathbf{x}} = \mathbf{A}^r\widehat{\mathbf{y}}$ where $\widehat{\mathbf{x}} = (1, \overline{\mathbf{x}})^t$ and $\widehat{\mathbf{y}} = (c, \overline{\mathbf{y}})$. Note that the equality of the first elements of $\widehat{\mathbf{x}}$ and $\mathbf{A}^r\widehat{\mathbf{y}}$ will provide Eq. (9); the full employment condition. Since the labor constraints are satisfied for $\widehat{\mathbf{y}}$ and that $\overline{\mathbf{A}}^r$ is a submatrix of $\mathbf{A}^r$, it follows that $\overline{\mathbf{x}} = \overline{\mathbf{A}}^r\overline{\mathbf{y}}$ holds.

$$
\begin{aligned}
\overline{\mathbf{z}} - \rho^{-1}\overline{\mathbf{x}} &= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}\{I + [\mathbf{I}-\overline{\Delta} - (1+g)\mathbf{G}\rho^{-1}]\overline{\mathbf{A}}^r\}\,\overline{\mathbf{y}} \\
&= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}I + \mathbf{I}-\Delta - (1+g)\begin{pmatrix} (1+a_1) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (1+a_n) \end{pmatrix} \\
&\qquad \left[\frac{(1+r)}{(1+g)(1+a_0)}\right]\Bigg]\,\overline{\mathbf{A}}^r\Bigg\}\,\overline{\mathbf{y}} \\
&= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}\left\{I + \left[\mathbf{I}-\Delta - (1+r)\begin{pmatrix} \frac{(1+a_1)}{(1+a_0)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \frac{(1+a_n)}{(1+a_0)} \end{pmatrix}\right]\overline{\mathbf{A}}^r\right\}\overline{\mathbf{y}} \\
&\geq \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}\{I + [\mathbf{I}-\Delta - (1+r)\mathbf{I}]\overline{\mathbf{A}}^r\}\,\overline{\mathbf{y}} \quad \text{due to Assumption 5,} \\
&= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}[\mathbf{I}-(r\mathbf{I}+\overline{\Delta})]\overline{\mathbf{A}}^r\overline{\mathbf{y}} > 0 \text{ from Assumption 4,}
\end{aligned}
$$

Therefore $\overline{\mathbf{y}}$ will be chosen so that $\overline{\mathbf{z}} - \rho^{-1}\overline{\mathbf{x}} \geq \mathbf{0}$ where $(\overline{\mathbf{x}}, \overline{\mathbf{z}})\varepsilon\mathbf{D}$. See also Lemma 3 through Lemma 7 in Takahashi (1985). ■

Remark 2 *It should be noted that since* $\widetilde{k}_i^\rho = \frac{k_i^\rho(t)}{A_i(t)L_t}$, *it follows that* $k_i^\rho(t) = \widetilde{k}_i^\rho A_i(t) =$

and further calculation will finally yield:

$$\mathbf{p}^{\rho}\left[-\mathbf{I}+\Delta+\left(\frac{1+r}{1+\alpha_0}\right)\mathbf{G}\right]=\mathbf{w}^{\rho}.$$

These are clearly non-arbitrage conditions among capital goods and imply that any capital good must yield the same rate of returns as the subjective discount rate ρ. Thus the Euler conditions are the non-arbitrage conditions.

Because of the differentiability and the constant returns to scale technologies, the well-known proposition proved by Samuelson (1945) will hold: the cost function denoted by $C^i(w_0, \mathbf{w}^{\rho})$ $(i = 1, \cdots, n)$ is homogeneous of degree one and $\partial C^i/\partial w_j = a_{ij}$ where $a_{ij} = k_{ij}/y_j$ $(i = 1, 2, \cdots, n;\ j = 0, 1, \cdots, n)$. Due to the cost minimization condition and this property, a unique technology matrix $\mathbf{A}^{\rho}$ is chosen along the OSS $\mathbf{k}^{\rho}$. Also note that due to Assumption 3, for a given $\rho \in (0, 1]$, the uniquely chosen technology matrix $\overline{\mathbf{A}}^{\rho}$ along the OSS $\mathbf{k}^{\rho}$ have to satisfy,

$$[\mathbf{I}-(r\mathbf{I}+\overline{\Delta})\overline{\mathbf{A}}^{\rho}]^{-1} \geq \Theta.$$

Furthermore it follows that $a^{\rho}_{00} > 0$ and $\mathbf{a}^{\rho}_{0.} \gg \mathbf{0}$ from Assumption 2. Henceforth, we use the symbol "ρ" to clarify that vectors and matrices are evaluated along OSS $\mathbf{k}^{\rho}$. Conbining these results, the following important property will be established:

Lemma 2. When $\rho \in (0, 1]$, there exists a unique OSS $(\mathbf{k}^{\rho} \gg \mathbf{0})$[6] with the

[6]Let $\mathbf{x}$ and $\mathbf{y}$ be n-dimensional vectors. Then $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for all i, $\mathbf{x} > \mathbf{y}$ if $x_i \geq y_i$ for all i and at least one j, $x_i > y_i$ and $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i$ for all i.

And the nonsingularity of $\mathbf{b}^\rho$ comes from the following observation: From Murata (1977), $\mathbf{b}^\rho = [\mathbf{a}^\rho - (1/a_{00}^\rho)\mathbf{a}_{\cdot 0}^\rho \mathbf{a}_{0\cdot}^\rho]^{-1}$. Furthermore, by Gantmacher (1960), it also follows that $det\ \mathbf{A}^\rho = a_{00}^\rho det[\mathbf{a}^\rho - (1/a_{00}^\rho)\mathbf{a}_{\cdot 0}^\rho \mathbf{a}_{0\cdot}^\rho]$. Since $\mathbf{A}^\rho$ is non-singular, the result follows.

From now on, we are concentrated on the OSS with $\rho = 1$ denoted by $\mathbf{k}^*$. We will also use " $*$ " to denote the elements and variables are evaluated at $\mathbf{k}^*$.

Definition. When $\rho = 1$, the chosen technology matrix $\mathbf{A}^*$ satisfies the *Generalized Capital Intensity GCI -I* condition, if there exists a set of positive number $(d_1, \cdots, d_n)$ such that

$$d_s\left(\frac{a_{ss}^*}{a_{0s}^*} - \frac{a_{s0}^*}{a_{00}^*}\right) > \sum_{i \neq s,0}^{n} d_i \left|\frac{a_{si}^*}{a_{0i}^*} - \frac{a_{s0}^*}{a_{00}^*}\right| \quad for\ s = 1, \cdots, n.$$

Similarly, the technology matrix $\mathbf{A}^*$ satisfies the *Generalized Capital Intensity GCI -II* condition, if there exists a set of positive number $(d_1, \cdots, d_n)$ such that

$$\frac{a_{ss}^*}{a_{0s}^*} - \frac{a_{s0}^*}{a_{00}^*} < 0$$

and

$$d_s\left|\frac{a_{ss}^*}{a_{0s}^*} - \frac{a_{s0}^*}{a_{00}^*}\right| > \sum_{i \neq s,0}^{n} d_i \left|\frac{a_{si}^*}{a_{0i}^*} - \frac{a_{s0}^*}{a_{00}^*}\right| \quad for\ s = 1, \cdots, n.$$

Consider a capital good sector s, and focus on its own capital input s and its capital-labor ratio in all the other sectors. By the definition the left-hand side of has a dominant diagonal that is positive (negative) for rows[7].

Proof. Due to Lemma 3, under the Strong GCI-I (the Strong GCI-II), the inverse matrix $\mathbf{B}^*$ has positive (negative) diagonal elements and negative (positive) off-diagonal elements. From the accumulation equation $\mathbf{y}^* = (1+n)\mathbf{G}\mathbf{k}^* - (\mathbf{I}-\Delta)\mathbf{k}^*$ and $\mathbf{y}^* = \mathbf{b}^*\mathbf{k}^* + \mathbf{b}^*_{.0}$, it follows that

$$[\mathbf{b}^* - ((1+g)\mathbf{G} + \Delta - \mathbf{I})]\mathbf{k}^* = -\mathbf{b}^*_{.0}$$

Due to Lemma 3, $-\mathbf{b}^*_{.0} < (>)\mathbf{0}$. Theirfore the matrix $[\mathbf{b}^* - ((1+g)\mathbf{G} + \Delta - \mathbf{I})]$ has the negative (positive) dominant diagonal for rows. ■

From now on we may call the dominant diagonal that is negative as the n.d.d. and also call the dominant diagonal that is positive as the p.d.d. for short.

From the Euler equations (12), its Jacobian $\mathbf{J}(\mathbf{k}, \rho)$ is

$$\mathbf{J}(\mathbf{k}, \rho) = \rho\mathbf{V}_{xx}(\mathbf{k}, \mathbf{k}) + \rho\mathbf{V}_{xz}(\mathbf{k}, \mathbf{k}) + \mathbf{V}_{zx}(\mathbf{k}, \mathbf{k}) + \mathbf{V}_{zz}(\mathbf{k}, \mathbf{k}),$$

which at $\mathbf{k}^*$ is

$$\mathbf{J}(\mathbf{k}, 1) = \mathbf{V}_{zz}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{xz}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{zx}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{xx}(\mathbf{k}^*, \mathbf{k}^*)$$

[7]Suppose $\mathbf{A}$ is an $n \times n$ matrix and its diagonal elements are negative (positive). Let there exist a positive vector $\mathbf{h}$ such that $h_i \mid a_{ii} \mid > \sum_{j=1, j\neq i}^{n} h_j \mid a_{ij} \mid$, $i = 1, 2, \cdots, n$. Then $\mathbf{A}$ is said to have a dominant main diagonal that is negative (positive) for rows. See McKenzie (1960) and Murata (1977).

where the suffix " t " means a transpose of a matrix. Utilizing these relations, all the partial derivative matrices at $\mathbf{k}^*$ can be expressed in terms of the matrices $\mathbf{b}^*$ and $\mathbf{T}_{22}$ as follows: $\mathbf{T}_{11} = (\mathbf{b}^*)^{-1}\mathbf{T}_{22}^t(\mathbf{b}^*)^{-1} = (\mathbf{b}^*)^{-1}\mathbf{T}_{22}(\mathbf{b}^*)^{-1}$, $\mathbf{T}_{12} = -(\mathbf{b}^*)^{-1}\mathbf{T}_{22}$, and $\mathbf{T}_{21} = -\mathbf{T}_{22}(\mathbf{b}^*)^{-1}$.Substituting $\mathbf{Y}_x = (g\mathbf{I}+\Delta)$ and $\mathbf{Y}_z = \mathbf{I}$ into Eq.(2.22) of Takahashi (1985), the Jacobian will be expressed as follows:

$$\mathbf{J}(\mathbf{k}^*, 1) = [(1+g)\mathbf{G}+\Delta-\mathbf{I}, \mathbf{I}] \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} (1+g)\mathbf{G}+\Delta-\mathbf{I} \\ \mathbf{I} \end{pmatrix}.$$

If the righthand side is negative definite, then the proof will be completed. Substituting all the relations obtained before into the Hessian matrix of the social production function and suppose that the matrix $\mathbf{b}^*$ is nonsingular, then we may yield the following equation:

$$\begin{aligned} & [(1+g)\mathbf{G}+\Delta-\mathbf{I}, \mathbf{I}] \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} (1+g)\mathbf{G}+\Delta-\mathbf{I} \\ \mathbf{I} \end{pmatrix} \\ = \; & ((1+g)\mathbf{G}+\Delta-\mathbf{I})\mathbf{T}_{11}((1+g)\mathbf{G}+\Delta-\mathbf{I}) \\ & +((1+g)\mathbf{G}+\Delta-\mathbf{I})\mathbf{T}_{21} + \mathbf{T}_{12}((1+g)\mathbf{G}+\Delta-\mathbf{I}) + \mathbf{T}_{22} \\ = \; & [\mathbf{b}^* - ((1+g)\mathbf{G}+\Delta-\mathbf{I})]^2[(\mathbf{b}^*)^{-1}]^2\mathbf{T}_{22}. \end{aligned}$$

Due to Lemma 4, the matrix $[\mathbf{b}^* - ((1+g)\mathbf{G}+\Delta-\mathbf{I})]$ has the negative (positive) d.d. from the GCI conditions and it must be nonsingular. $\mathbf{b}^*$ is also nonsingular. $\mathbf{T}_{22}$

From the definition above, the NMF is a set of $(\mathbf{x}, \mathbf{z})$ capital stock vectors which arise from the exact same net benefit as that of OSS when it is evaluated by the prices of OSS. Also, the VMF is the projection of a flat on the surface of the utility function V that is supported by the price vector $(-\mathbf{p}^\rho, \rho\mathbf{p}^\rho, 1)$ onto the $(\mathbf{x}, \mathbf{z})$-space. In Takahashi (1985), I consider the case of the objective function where n capital goods as well as pure-consumption goods are also consumable. Here, the capital goods are not consumable but the discounted sum of the sequence of pure-consumption goods is directly evaluated. Due to the well-established Nonsubstitution Theorem, along the OSS, a unique technology matrix $\mathbf{A}^\rho$ defined before will be chosen.

By exploiting this fact, we will re-characterize the VMF as a more tractable formula with the (n+1) by (n+1) matrix $\mathbf{A}^\rho$ and (n+1)-dimensional vectors as follows:

Lemma 6. When $\mathbf{A}^\rho$ is non-singular, $(\mathbf{x}, \mathbf{z}) \in \mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$ if and only if there exists $\widehat{\mathbf{y}} \equiv (c, \mathbf{y})' \geqq 0$ such that

$$i)\ \widehat{\mathbf{x}} = \mathbf{A}^\rho \widehat{\mathbf{y}}$$

$$ii)\ \widehat{\mathbf{z}} = \left(\frac{1}{1+n}\right) \overline{\mathbf{G}}^{-1}[\widehat{\mathbf{y}} + (\mathbf{I} - \overline{\Delta})]\widehat{\mathbf{x}}$$

$\{[\mathbf{p}^\rho \mathbf{y} - \mathbf{w}^\rho \mathbf{x}] - [\mathbf{p}^\rho \mathbf{y}^\rho - \mathbf{w}^\rho \mathbf{k}^\rho]\} = 0$ and $c = c^\rho$. Thus it finally yields that

$$c + \mathbf{p}^\rho \mathbf{y} - \mathbf{w}^\rho \mathbf{x} = c^\rho + \mathbf{p}^\rho \mathbf{y}^\rho - \mathbf{w}^\rho \mathbf{k}^\rho.$$

This result is Condition i), which implies that $(c_0, \mathbf{y}, \mathbf{x})$ should lie on the production frontier of $T(\mathbf{y}, \mathbf{k})$ and in each sector, the chosen technology must be the same as that in the OSS. In other words, the OSS technology matrix $\mathbf{A}^\rho$ will be chosen. Thus on the NMF, the exact same technology matrix as the corresponding OSS is chosen. In other words, given OSS technology matrix $\mathbf{A}^\rho$, the cost minimization and the full-employment conditions for labor and capital goods are satisfied. Therefor, the following equations must hold:

$$1) q^\rho = w_0^\rho a_{00}^\rho + \mathbf{w}^\rho \mathbf{a}_{\cdot 0}^\rho,$$

$$2)\ \mathbf{p}^\rho = w_0^\rho a_{0\cdot}^\rho + \mathbf{w}^\rho \mathbf{a}^\rho,$$

$$3) 1 = a_{00}^\rho c + \mathbf{a}_{0\cdot}^\rho \mathbf{y},$$

$$4)\ \mathbf{x} = a_{\cdot 0}^\rho c + \mathbf{a}^\rho \mathbf{y}$$

The cost-minimization conditions 1) and 2) imply that the same technology as that of OSS is chosen. 3) and 4) means that, under the chosen technology, the full employment conditions hold. It is not difficult to see that 3) and 4) can be summarized as Condition ii). From these conditions, it follows that $c(t) > 0$ and $\mathbf{y}(t) \gg \mathbf{0}$ for all t, respectively. Condition ii) are the (n+1)-dimensional capital accumulation equations and $\mathbf{z}$ is determined through this relation. ■

independent vectors $\mathbf{d}^h$ $(h = 1, \cdots, n-1)$. It is clear that $\mathbf{d}^h$ shows a reallocation of fixed labor among sectors. Moreover, define the following: for $h = 1, 2, \cdots, n$ and a positive scalar ε_h,

$$
\begin{aligned}
\widehat{\mathbf{y}}^h &\equiv \widehat{\mathbf{y}}^\rho + \varepsilon_h \mathbf{d}^h, \\
\widehat{\mathbf{x}}^h &\equiv \mathbf{A}^\rho \widehat{\mathbf{y}}^h = \mathbf{A}^\rho \widehat{\mathbf{y}}^\rho + \varepsilon_h \mathbf{A}^\rho \mathbf{d}^h \\
&= \widehat{\mathbf{k}}^\rho + \varepsilon_h \mathbf{A}^\rho \mathbf{d}^h
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{\mathbf{z}}^h &\equiv \overline{\mathbf{G}}^{-1}[\widehat{\mathbf{y}}^h + (\mathbf{I} - \overline{\Delta})\widehat{\mathbf{x}}^h] \\
&= \overline{\mathbf{G}}^{-1}[\widehat{\mathbf{y}}^\rho + (\mathbf{I} - \overline{\Delta})\widehat{\mathbf{x}}^\rho] + \varepsilon_h \overline{\mathbf{G}}^{-1}[\mathbf{d}^h + (\mathbf{I} - \overline{\Delta})\mathbf{A}^\rho \mathbf{d}^h] \\
&= \widehat{\mathbf{k}}^\rho + \varepsilon_h \overline{\mathbf{G}}^{-1}[\mathbf{I} + (\mathbf{I} - \overline{\Delta})\mathbf{A}^\rho]\mathbf{d}^h
\end{aligned}
$$

Note that the first element of the vector $\mathbf{A}^\rho \mathbf{d}^h$ is zero due to the fact that $\sum_{i=0}^n a_{0i} d_i^h = 0$ for all h. Since the first element of $\widehat{\mathbf{k}}^\rho$ is one, the first element of $\widehat{\mathbf{x}}^h$ will be one. So the vectors $\widehat{\mathbf{x}}^h$ $(h = 1, \cdots, n)$ are well defined. Since the first element of $\widetilde{\mathbf{k}}^\rho$ is 1, $\widetilde{\mathbf{z}}^h$ is also well defined for all h. due to the fact that $\widehat{\mathbf{y}}^\rho \gg \mathbf{0}$ and $\widehat{\mathbf{k}}^\rho \gg \mathbf{0}$, ε_h can be chosen so that $\widehat{\mathbf{y}}^h > \mathbf{0}$,$\widehat{\mathbf{x}}^h > \mathbf{0}$ and $\widehat{\mathbf{z}}^h > \mathbf{0}$ for all h. From our way of construction, the vectors $\widehat{\mathbf{y}}^h$,$\widehat{\mathbf{x}}^h$ and $\widehat{\mathbf{z}}^h$ satisfy Lemma 2 and the corresponding vector $(\mathbf{x}^h, \mathbf{z}^h)$ also belongs to $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$ for all h. This implies that there are n linearly independent vectors $(\mathbf{x}^h - \mathbf{k}^\rho, \mathbf{z}^h - \mathbf{k}^\rho)$. Therefore, there are exactly n linearly independent line

Proof. See the argument of Section 4 of Takahashi (1993). ■

The Neighborhood Turnpike Theorem means that any optimal path must be trapped in a neighborhood of the corresponding OSS and the neighborhood can be taken as small as possible by making ρ close enough to one.

5 Turnpike Theorem

The full Turnpike Theorem is described as the following theorem:

Full Turnpike Theorem There is a $\overline{\rho} > 0$ close enoug to 1 such that for any $\rho \in [\overline{\rho}, 1)$, an optimal path $\mathbf{k}^{\rho}(t)$ with the sufficient initial capital stock will asymptotically converge to the optimal steady state $\mathbf{k}^{\rho}$.

As we have shown, under Assumption 7, the dimension of the VMF is n. We will keep this assumption henceforth. On the other hand, the dynamics of the VMF is expressed by the n-dimensional linear difference equation (8). To show the full turnpike theorem we need to strengthen the generalized capitl intensity conditionds, GCI-I and GCI-II.

Remark 3 *the first to be noted that in the efficiency unit term, the full turnpike means that each sector's optimal path converges to the optimal steady state. In original terms of series, any industry's per-capita capital stock and output grow at the rate of*

Lemma 10. Under the negative (positive) d.d., the n-dimensional NMF, $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$ where $\rho \in [\overline{\rho}, 1)$ turns out to be a linear stable (unstable) manifold.

Proof. Because $\mathbf{b}^\rho + \mathbf{I} - \Delta = [\mathbf{b}^\rho + (\mathbf{I} - \Delta) - (1+g)\mathbf{G}] + (1+n)\mathbf{G}$, it follows that $(1/(1+g)\mathbf{G}[\mathbf{b}^\rho + \mathbf{I} - \Delta] = (1/(1+g))\mathbf{G}[\mathbf{b}^\rho + (\mathbf{I} - \Delta) - (1+n)\mathbf{G}] + \mathbf{I}$. Defining $\mathbf{C} = (1/(1+g))\mathbf{G}[\mathbf{b}^\rho + (\mathbf{I} - \Delta) - (1+g)\mathbf{G}]$, Eq.(8) can be rewritten as:

$$\boldsymbol{\eta}(t+1) = (\mathbf{C} + \mathbf{I})\boldsymbol{\eta}(t).$$

Note again that $\boldsymbol{\eta}(t) = (\mathbf{x} - \mathbf{k}^\rho)$ and $\boldsymbol{\eta}(t+1) = (\mathbf{z} - \mathbf{k}^\rho)$. Thus applying Lemma 4, under the negative d.d. (the positive d.d.), any path on NMF will converge to (diverge from) the OSS. ■

From this lemma, under the Strong GCI-II condition, the local stability and the stability of the NMF hold simultaneously. The stability of the NMF implies that the Neighborhood turnpike holds. Thus combining both results, the following full Turnpike theorem will be established.

Corollary. Under the Strong GCI-II condition, the full Turnpike Theorem will be established.

Proof. To achieve the full Turnpike theorem, we need to combin the Neighborhood Turnpike Theorem and the local stability of the OSS. The Neighborhood Turnpike Theorem means that any optimal path should be trapped in the neighborhood of the

$$\det \mathbf{V}^{\rho}_{xz} = -(\det(\mathbf{b}^{\rho})^{-1})^2 \det[\mathbf{b}^{\rho} + (\mathbf{I} - \Delta)] \det \mathbf{T}^{\rho}_{22}$$

Since $\mathbf{T}^{\rho}_{22}$ is negative-definite, it is non-singular. Furthermore, $[\mathbf{b}^{\rho} + (\mathbf{I} - \Delta)]$ has a quasi-dominant diagonal that is positive under the GCI-I condition, it also non-singular. Thus $\mathbf{V}^{\rho}_{xz}$ is non-singular. On the other hand, the GCI-I condition implies that the VMF is explosive. This means that there are n characteristic roots with absolute value greater than one. Applying Lemma 11, this also implies that there are n characteristic roots with its absolute value less than one. So the OSS satisfies the local stability. ∎

Thus we have established the following theorem:

Theorem 2 Under the both GCI conditions, the OSS $\mathbf{k}^{\rho}$ exhibits the full Turnpike Theorem.

Proof. Under the GCI-II condition, the full Turnpike Theorem will be established due to the above corollary. On the other hand, under the GCI-I condition, from Lemma 12, the OSS will exhibit the local stability. Since any path on the VMF is totally unstabl, the NMF is "stable" and the Neighborhood Turnpike Theorem hold. Combining both results again, the full Turnpike Theorem is also established. This completes the proof. ∎

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$\mathbf{F}(\mathbf{k}(\rho'))$ where $(\mathbf{x}', \mathbf{z}') \in int\ D$, $(\mathbf{x}, \mathbf{z}) \neq (\mathbf{x}', \mathbf{z}')$ and $\rho' \in [\overline{\rho}, 1)$ is chosen close enough to ρ. Now let us define the plain $\mathbf{H}^\alpha$ is defined as follows:

$$\mathbf{H}^\alpha \equiv \{(\mathbf{x}, \mathbf{z}) \in D : t_0[\alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho), \alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho)]+$$

$$\sum_{h=1}^{n} t_h[\alpha\mathbf{x}^j(\rho') + (1-\alpha)\mathbf{x}^j(\rho), \alpha\mathbf{z}^j(\rho') + (1-\alpha)\mathbf{z}^j(\rho)]\} \ where \ \sum_{h=1}^{n} t_h = 1.$$

We can always find an intersection $(\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ between $\mathbf{H}^\alpha$ and the line obtained by connecting points $(\mathbf{x}, \mathbf{z})$ and $(\mathbf{x}', \mathbf{z}')$ unless $(\mathbf{x}, \mathbf{z}) = (\mathbf{x}', \mathbf{z}')$. Since $(\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ is on the plain $\mathbf{H}^\alpha$, it can also be expressed as follows:

$$(\mathbf{x}^\alpha, \mathbf{z}^\alpha) = t_0^\alpha[\alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho), \alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho)]+$$

$$\sum_{h=1}^{n} t_h^\alpha[\alpha\mathbf{x}^j(\rho') + (1-\alpha)\mathbf{x}^j(\rho), \alpha\mathbf{z}^j(\rho') + (1-\alpha)\mathbf{z}^j(\rho)] \ where \ \sum_{h=1}^{n} t_h^\alpha = 1.$$

where $\alpha \to 0$, $[\alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho)] \to \mathbf{k}(\rho)$ and $[\alpha\mathbf{x}^h(\rho') + (1-\alpha)\mathbf{x}^h(\rho), \alpha\mathbf{z}^h(\rho') + (1-\alpha)\mathbf{z}^h(\rho)] \to (\mathbf{x}^\rho, \mathbf{z}^\rho)$ for $h = 1, \cdots, n$. Therefore $(\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ converges to $(\mathbf{x}, \mathbf{z})$ as $\alpha \to 0$. Also note that $(\mathbf{x}^\alpha, \mathbf{z}^\alpha) \in int\ D$ due to the convexity of D and the fact that $(\mathbf{x}', \mathbf{z}') \in int\ D$. On the other hand, because of the continuity of $\mathbf{k}(\rho), \mathbf{x}(\rho)$ and $\mathbf{z}(\rho)$ in $\rho \in [\overline{\rho}, 1)$. for any $\varepsilon^\alpha > 0$ there exists $\delta^\alpha > 0$ such that $|\rho^\alpha - \rho| < \delta^\alpha$ implies that

$$||\mathbf{k}(\rho^\alpha) - [\alpha\mathbf{k}(\rho') + (1-\alpha)\mathbf{k}(\rho)]|| < \varepsilon^\alpha$$

and for $h = 1, \cdots, n$,

$$\left|\left|(\mathbf{k}(\rho^\alpha), \mathbf{k}(\rho^\alpha)) - [\alpha\mathbf{x}^h(\rho') + (1-\alpha)\mathbf{x}^h(\rho), \alpha\mathbf{z}^h(\rho') + (1-\alpha)\mathbf{z}^h(\rho)]\right|\right| < \varepsilon^\alpha$$

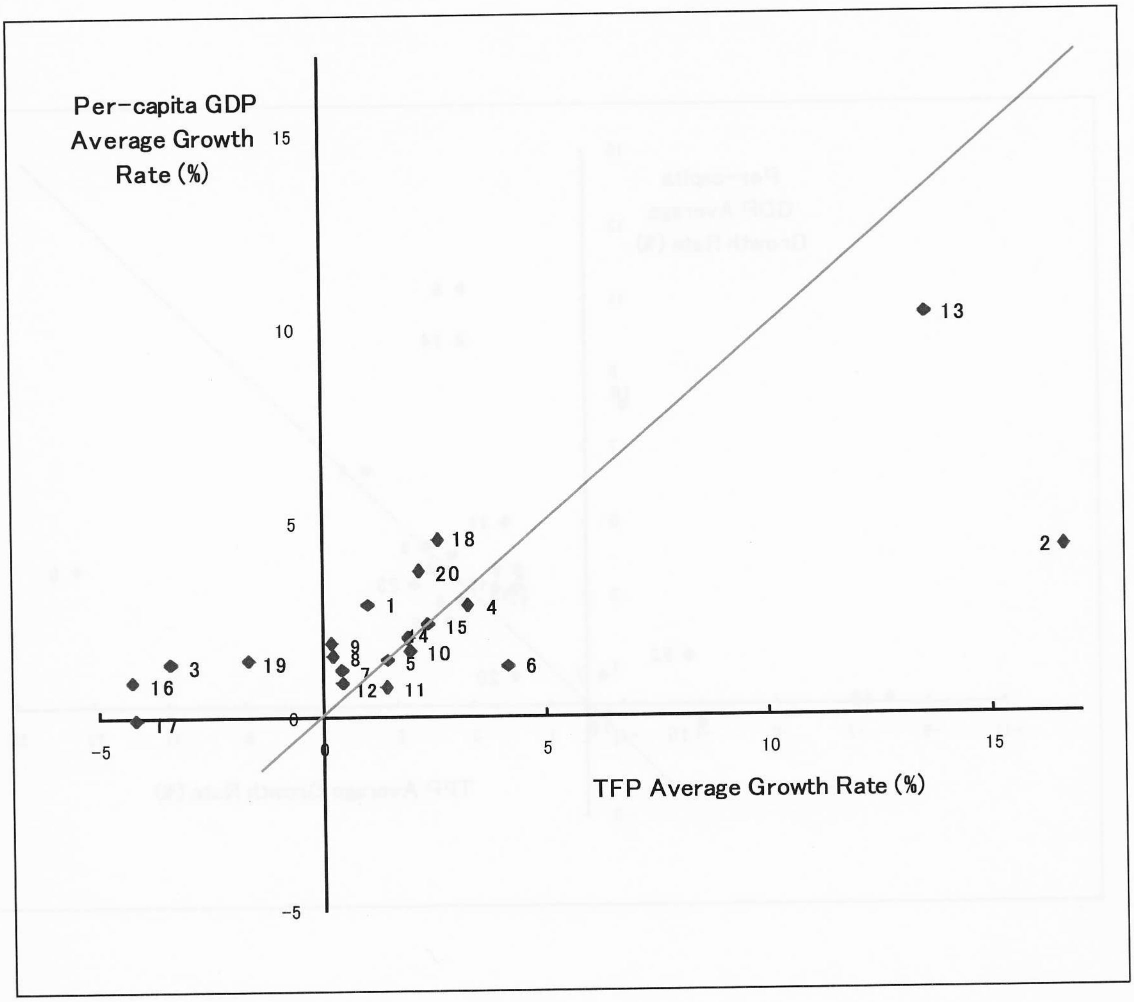

Figure 1: U.S. Economy, 1970-2005 Source: EU-KLEMS DATABASE