# Competitive Equilibria of a Large Exchange Economy on the Commodity Space $\ell^\infty$

Mitsunori Noguchi[1] Takashi Suzuki[2]

April 2009; Revised, March 2010

[1]Department of Economics, Meijo University, Tengoya 1-501 Shiogamaguchi, Tenpaku-ku, Nagoya 468, Japan

[2]Department of Economics, Meiji-Gakuin University, 1-2-37 Shiroganedai, Minato-ku, Tokyo, 108, Japan

Abstract

The existence of competitive equilibrium for a large exchange economy over the commodity space $\ell^\infty$ will be discussed. We define the economy as a distribution on the space of consumers' characteristics following Hart, Hildenbrand and Kohlberg (1974), and proved the theorem without the convexity of preferences. The case in which the indivisible commodities present will also be discussed.

JEL classification: D51

# 1 Introduction

In this paper, we are concerned with an exchange economy with a continuum of consumers introduced by Aumann (1966) and with an infinite time horizon introduced by Bewley (1970 and 1991), respectively. The economy is formulated on the commodity space $\ell^\infty$,

$$\ell^\infty = \left\{ \boldsymbol{x} = (x^t) |\ sup_{t \geq 1} |x^t| < +\infty \right\},$$

the space of the sequences with bounded supremum norms. It is well known that the space $\ell^\infty$ is a Banach space with respect to the norm $\|\boldsymbol{x}\| = sup_{t \geq 1} |x^t|$ for $\boldsymbol{x} \in \ell^\infty$ (Royden (1988)). We will deal with an exchange economy throughout this paper, hence there exist no producers in the economy.

It is well known that the dual space of $\ell^\infty$ is the space of bounded and finitely additive set functions on $\mathbb{N}$ which is denoted by $ba$,

$$ba = \left\{ \pi : 2^{\mathbb{N}} \to \mathbb{R} \middle| sup_{E \subset \mathbb{N}} |\pi(E)| < +\infty,\ \pi(E \cup F) = \pi(E) + \pi(F) \right.$$
$$\left. \text{whenever } E \cap F = \emptyset \right\}.$$

Then we can show that the space $ba$ is a Banach space with the norm

$$\|\pi\| = sup \left\{ \sum_{i=1}^{n} |\pi(E_i)| \middle|\ E_i \cap E_j = \emptyset \text{ for } i \neq j, n \in \mathbb{N} \right\}.$$

Since the commodity vectors are represented by sequences, it is more natural to consider the price vectors also as sequences rather than the set functions. Therefore the subspace $ca$ of $ba$, which is the space of the bounded and countably additive set functions on $\mathbb{N}$ is more appropriate as the price space. Indeed it is easy to see that the space $ca$ is isometrically isomorphic to the space $\ell^1$, the space of all summable sequences,

$$\ell^1 = \left\{ \boldsymbol{p} = (p^t) \middle|\ \sum_{t=1}^{\infty} |p^t| < +\infty \right\},$$

which is a Banach space with the norm

$$\|\boldsymbol{p}\| = \sum_{t=1}^{\infty} |p^t|.$$

Then the value of a commodity $\boldsymbol{x} = (x^t) \in \ell^\infty$ evaluated by a price vector $\boldsymbol{p} = (p^t) \in \ell^1$ is given by the natural "inner product" $\boldsymbol{px} = \sum_{t=1}^{\infty} p^t x^t$.

The exchange economy with a measure space $(A, \mathcal{A}, \lambda)$ of consumers was first introduced by Aumann (1966) on a finite dimensional commodity space. As is well known, he defined the economy by a measurable map $\mathcal{E} : A \to \mathcal{P} \times \Omega$. Each element $a \in A$ is interpreted as a "name" of a consumer, and each value of the map $\mathcal{E}(a) = (\succsim_a, \omega_a)$ is the characteristics of the consumer $a$. He established the existence of the competitive equilibrium of such an economy and observed that the convexity assumption on the preference relations of the consumers is not necessary. This is a consequence of Liapunov's theorem which asserts that on the nonatomic measure space, the integration of a measurable correspondence is a convex set. This means that even if the demand of an individual consumer is not convex valued, the total demand which is defined by the integration over the set $A$ of the individual demand correspondence is convex valued. In the course of the proof of the existence theorem, the $\ell$-dimensional version of Fatou's lemma was essentially used.

Several authors have tried to unify the above results of Aumann (1966) and Bewley (1970). For example, Khan and Yannelis (1991) and Noguchi (1997a) proved the existence of a competitive equilibrium for the economies with a measure space of agents in which the commodity space is a separable Banach space whose positive orthant has an (norm) interior point[1]. Bewley (1991) and Noguchi (1997b) proved the equilibrium existence theorems for the economies with a measure space of consumers on the commodity space $\ell^\infty$. Bewley worked with an exchange economies, and Noguchi (1997a and b) proved his theorems for the economies with continuum of consumers and producers.

These authors formulated their economies and the concept of competitive equilibria as similar as Aumann did, namely that the economies are measurable maps from the set $A$ of agents to the set of the characteristics and the allocations are measurable maps from the set $A$ to the commodity space (the parametric approach or the micro-economic approach). As a consequence, they have recognized on their works that there are significant technical di culties for extensions of the Aumann's theorem to infinite dimensional commodity spaces. For examples, the Liapunov's theorem and the Fatou's lemma only hold in a very restricted manner. Therefore they had to pay expensive costs which Aumann could dispense with. Khan and Yannelis, and Bewley assumed that the preferences are convex. Noguchi assumed that a commodity vector does not belong to the convex hull of its preferred set. These assumptions obviously weaken the impact of the Aumann's classical result which revealed the "convexfying e ect" of large numbers of the economic agents.

We define our economy as a probability measure $\mu$ on the set of agents' characteristics $\mathcal{P} \times \Omega$, where $\mathcal{P}$ is the set of preferences and $\Omega$ is the set of initial endowments. Then the competitive equilibrium of this economy is also defined as a probability measure $\nu$ on $X \times \mathcal{P} \times \Omega$, where the set $X \subset \ell^\infty$ is a consumption set which is assumed to be identical among all consumers (the distribution approach or the macro-economic approach).

These definitions of the economy and the competitive equilibrium on it were first proposed by Hart, Hildenbrand and Kohlberg (1974), and was applied by Mas-Colell (1975) and Jones (1983) for the model with the commodity space $ca(K)$, the set of (signed) measures on a compact metrics space $K$ (the model of commodity di erentiations). We can interpret $(\boldsymbol{x}, \succsim, \omega) \in support(\nu)$

[1]Since the space $\ell^\infty$ is not separable, these results are not considered as generalizations of Bewley (1970).

in such a way that $\boldsymbol{x} \in X$ is the allocation assigned to $\nu_{\mathcal{P}\times\Omega}$ percent[2] of consumers with the characteristics $(\succsim, \omega) \in \mathcal{P}\times\Omega$. The main result of this paper is the existence of competitive equilibrium distribution for a given economy (Theorem 1).

Technically speaking, the distribution approach is weaker than the parametric approach of Aumann in the following sense. Aumann finds an allocation $f : A \to X$ (and an equilibrium price vector) for any arbitrarily given measurable map (economy) $\mathcal{E} : A \to \mathcal{P} \times \Omega$. However, given an economy $\mu$, certainly we can find an equilibrium distribution $\nu$ (and an equilibrium price vector), and by the Skorokhod's theorem (see Fact 5), there exist measurable maps $\mathcal{E} : A \to \mathcal{P} \times \Omega$ and $f : A \to X$ such that $\nu = \lambda \circ (f, \mathcal{E})^{-1}$. But it is not true that we can find for an arbitrarily given map $\mathcal{E} : A \to \mathcal{P} \times \Omega$, a measurable map $f : A \to X$ such that $\nu \equiv \lambda \circ (f, \mathcal{E})^{-1}$ is an equilibrium distribution.

On the other hand, however, from the economic point of view, what we are really interested in is the performance of the market itself rather than the behavior of each individual. For this it is enough to know the distribution of consumers' characteristics, and we do not have to know who has which character. In other words, even if the economy is defined by the distribution $\mu$ rather than the map $\mathcal{E}$, almost nothing is lost from the point of view of economic theorists and/or policy makers. The philosophy which emphasizes the distribution more than each individual was already addressed by Hildenbrand (1974).

We claim that the distribution approach has at least two important advantages. First, it simplifies the proof. This seems to be obvious from the discussions of next section when they are compared to those of the above authors. More importantly, this approach does not use the Liapunov's theorem nor the Fatou's lemma. Therefore we do not have to assume any convexity-like assumptions on the consumers' preferences. On account of these nice characters, the distribution approach seems to make itself to be seriously taken and systematically exploited in the research of the market models with a continuum of agents and an infinite dimensional commodity space.

The next section presents a model and the main result. Section 3 discusses a model which includes the indivisible commodities using the powerful result of Mas-Colell (1977). Section 4 concludes the paper by giving some remarks on the other literatures of this subject.

[2] $\nu_{\mathcal{P}\times\Omega}$ is the marginal distribution of $\nu$ on $\mathcal{P} \times \Omega$.

## 2 The Model and Result

The set function $\pi \in ba$ is called purely finitely additive if $\rho = 0$ whenever $\rho \in ca$ and $0 \leq \rho \leq \pi$. The relation between the $ba$ and $ca$ is made clear by the next fundamental theorem,

Fact 1 (Yosida-Hewitt). If $\pi \in ba$ and $\pi \geq 0$, then there exist set functions $\pi_c \geq 0$ and $\pi_p \geq 0$ in $ba$ such that $\pi_c$ is countably additive and $\pi_p$ is purely finitely additive and satisfy $\pi = \pi_c + \pi_p$. This decomposition is unique.

On the space $\ell^\infty$, we can consider the several topologies. One is of course the norm topology $\tau_{norm}$ which was explained above. It is the strongest topology among the topologies which appear in this paper.

The weakest topology in this paper is the product topology $\tau_d$ which is induced from the metric

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{t=1}^{\infty} \frac{|x^t - y^t|}{2^t(1 + |x^t - y^t|)} \text{ for } \boldsymbol{x} = (x^t), \boldsymbol{y} = (y^t) \in \ell^\infty.$$

The product topology is nothing but the topology of coordinate-wise convergence, or $\boldsymbol{x} = (x^t) \to \mathbf{0}$ if and only if $x^t \to 0$ for all $t \in \mathbb{N}$.

A net $(\boldsymbol{x}_\alpha)$ on $\ell^\infty$ is said to converge to $\mathbf{0}$ in the weak* topology or $\sigma(\ell^\infty, \ell^1)$-topology if and only if $\boldsymbol{p}\boldsymbol{x} \to 0$ for each $\boldsymbol{p} \in \ell^1$. The weak* topology is characterized by the weakest topology on $\ell^\infty$ which makes $(\ell^\infty)^* = \ell^1$. Then it is stronger than the product topology, since the latter is characterized by $\boldsymbol{x}_\alpha \to 0$ if and only if $\boldsymbol{e}_t\boldsymbol{x} \to 0$ for all for each $\boldsymbol{e}_t = (0 \dots 0, 1, 0 \dots) \in \ell^1$, where $1$ is in the $t$-th coordinate.

The strongest topology on $\ell^\infty$ which makes $(\ell^\infty)^* = \ell^1$ is called the Mackey topology $\tau(\ell^\infty, \ell^1)$. It is characterized by saying that a net $(\boldsymbol{x}_\alpha)$ on $\ell^\infty$ is said to converge to $\mathbf{0}$ in $\tau(\ell^\infty, \ell^1)$-topology if and only if $sup\{|\boldsymbol{p}\boldsymbol{x}_\alpha||\boldsymbol{p} \in C\} \to 0$ on every $\sigma(\ell^1, \ell^\infty)$-compact, convex and circled subset $C$ of $\ell^1$, where a set $C$ is circled if and only if $rC \subset C$ for $-1 \leq r \leq 1$, and the topology $\sigma(\ell^1, \ell^\infty)$ is defined analogously as $\sigma(\ell^\infty, \ell^1)$, namely that a net $(\boldsymbol{p}_\alpha)$ on $\ell^1$ is said to converge to $\mathbf{0}$ in the $\sigma(\ell^1, \ell^\infty)$-topology if and only if $\boldsymbol{p}\boldsymbol{x} \to 0$ for each $\boldsymbol{x} \in \ell^\infty$. The topology $\tau(\ell^\infty, \ell^1)$ is weaker than the norm topology. Hence we have $\tau_d \subset \sigma(\ell^\infty, \ell^1) \subset \tau(\ell^\infty, \ell^1) \subset \tau_{norm}$.

Similarly, a net $(\pi_\alpha)$ on $ba$ is said to converge to $\mathbf{0}$ in the weak* topology or $\sigma(ba, \ell^\infty)$-topology if and only if $\pi\boldsymbol{x} \to 0$ for each $\boldsymbol{x} \in \ell^\infty$.

We now describe our economy in this paper. Let $\beta > 0$ be a given positive number, and $\ell$ be a positive integer. We will assume that the consumption set $X$ of each consumer is the set of nonnegative vectors whose coordinates after $\ell$ are bounded by $\beta$,

$$X = \{\boldsymbol{x} = (x^t) \in \ell^\infty | \ 0 \leq x^t \text{ for } t \geq 1, \ x^t \leq \beta \text{ for } t > \ell\}.$$

Of course the $\beta > 0$ is intended to be a very large number. We call the first $\ell$ commodities, $x^1, x^2 \dots x^\ell$ the primary commodities. Then it is obvious that the consumption set is written as

$$X = P \times Z,$$

where $P = \mathbb{R}^\ell_+$ and $Z = \{\boldsymbol{x} = (x^t) \in \ell^\infty |\ 0 \le x^t \le \beta$ for all $t \ge 1\}$. From now on, we will sometimes denote $\boldsymbol{x} = (\boldsymbol{y}, \boldsymbol{z}) \in P \times Z$ for $\boldsymbol{x} \in X$. We can use the next proposition on bounded subsets of $\ell^\infty$.

Fact 2 (Bewley (1991 a, p.226)). Let $Z$ be a (norm) bounded subset of $\ell^\infty$. Then on the set $Z$, the Mackey topology $\tau(\ell^\infty, \ell^1)$ coincides with the product topology $\tau_d$.

Hence on the set $Z$, we have $\tau_d = \sigma(\ell^\infty, \ell^1) = \tau(\ell^\infty, \ell^1)$. Moreover the bounded subsets of $\ell^\infty$ are $\sigma(\ell^\infty, \ell^1)$-weakly compact, namely that the weak* closure of the sets are weak*-compact by the Banach-Alaoglu's theorem,

Fact 3 (Alaoglu). If $X$ is a normed space, then the unit ball of $X^*$, $B = \{\boldsymbol{p} \in X^* | \|\boldsymbol{p}\| \le 1\}$ is compact in the $\sigma(X^*, X)$-topology.

Obviously $Z$ is $\tau_d$-closed, hence $\sigma(\ell^\infty, \ell^1)$-compact. This type of a consumption set already appeared in Mas-Colell (1975) in which the consumption set was assumed to be $\mathbb{R}_+ \times M$, where $M$ is a bounded subset of $\mathcal{M}(K)$, the space of measures on a compact metric space $K$. He also assumed that the measures in $M$ are integer valued, and an element of $\mathbb{R}_+$ the homogeneous good. We will discuss a similar situation in the next section.

We denote the set of all closed subsets of a set $S$ by $\mathcal{F}(S)$. The topology $\tau_c$ on $\mathcal{F}(S)$ of closed convergence is a topology which is generated by the base

$$[K_i; G_1 \dots G_n] = \{F \in \mathcal{F}(S) | F \cap K = \emptyset, F \cap G_i \ne \emptyset, i = 1 \dots n\}$$

as $K$ ranges over the compact subsets of $S$ and $G_i$ are arbitrarily finitely many open subsets of $S$. It is well known that if $X$ is locally compact separable metric space, then $\mathcal{F}(X)$ is compact and metrizable. Moreover, a sequence $F_n$ converges to $F \in \mathcal{F}(S)$ if and only if $Li(F_n) = F = Ls(F_n)$, where $Li(F_n)$ denotes the topological limes inferior of $\{F_n\}$ which is defined by

$\boldsymbol{x} \in Li(F_n)$ if and only if there exists an integer $N$ and a sequence $\boldsymbol{x}_n \in Li(F_n)$ for all $n \ge N$ and $\boldsymbol{x}_n \to \boldsymbol{x}$ $(n \to \infty)$,

and $Ls(F_n)$ is the topological limes superior which is defined by

$\boldsymbol{x} \in Ls(F_n)$ if and only if there exists a sequence $F_{n_q}$ with $\boldsymbol{x}_{n_q} \in F_{n_q}$ for all $q$ and $\boldsymbol{x}_{n_q} \to \boldsymbol{x}$ $(q \to \infty)$,

see Hildenbrand (1974, pp.15-19). Since $Z$ is compact in $\tau_d$ (hence $\sigma(\ell^\infty, \ell^1)$ and $\tau(\ell^\infty, \ell^1)$) topology, $X = \mathbb{R}^\ell \times Z$ is locally compact separable metric space. Hence $\mathcal{F}(X \times X)$ is a compact metric space, so that it is complete and separable.

Let $\mathcal{P} \subset \mathcal{F}(X \times X)$ be the collection of allowed preference relations which will be assumed to be compact. We will make the following assumptions concerning preferences of $\mathcal{P}$:

$(i)$ $\succsim \in \mathcal{P}$ is complete, transitive and reflexive,

$(ii)$ (local non-satiation) for each $\boldsymbol{x} \in X$ and every neighborhood $U$ of $\boldsymbol{x}$, there exists $\boldsymbol{w} \in U$ such that $\boldsymbol{x} \prec \boldsymbol{w}$, where $\boldsymbol{x} \prec \boldsymbol{w}$ means that $(\boldsymbol{x}, \boldsymbol{w}) \notin \precsim$,

$(iii)$ (overriding desirability of the primary commodities) for each $\boldsymbol{x} = (x^t)$ and $\boldsymbol{w} = (w^t) \in X$ and every $t = 1 \dots \ell$, there exists $\delta > 0$ such that $\boldsymbol{x} \prec \boldsymbol{w} + \delta \boldsymbol{e}_t$,
where $\boldsymbol{e}_t = (0 \dots 0, 1, 0 \dots)$ and $1$ is in the $t$-th coordinate. Note that preferences are $\tau_d$ (hence $\sigma(\ell^\infty, \ell^1)$ and $\tau(\ell^\infty, \ell^1)$) continuous, since $\mathcal{P} \subset \mathcal{F}(X \times X)$. Hence nearby commodities are considered to be uniformly (since $\mathcal{P}$ is compact) good substitutes.

An initial endowment is assumed to be nonnegative vectors $\omega$ of $\ell^\infty$, or $\omega \in \ell^\infty_+$. We will restrict the set of $\Omega$ of all allowed endowments is of the form

$$\Omega = \{\omega = (\omega^t) \in \ell^\infty |\ 0 \le \omega^t \le \gamma,\ \ t \in \mathbb{N}\}$$

for some fixed positive number $\gamma < \beta$. It is $\tau_d$-compact subset of $\ell^\infty$, hence $\sigma(\ell^\infty, \ell^1)$-compact by the same reason of the set $Z$.

Let $(K, \mathcal{B}(K), \xi)$ be a measure space where $K$ is a compact metric space and $\xi$ is a Borel probability measure on $(K, \mathcal{B}(K))$. A map $f : K \to \ell^\infty$ is said to be Pettis integrable if and only if for each $\pi \in ba$, $\pi f(a)$ is an integrable function on $(K, \mathcal{B}(K), \xi)$ and that there exists an element $\int f(a) d\xi \in \ell^\infty$ such that for each $\pi \in ba$, $\pi \int f(a) d\xi = \int \pi f(a) d\xi$. Then we have

Fact 4 (Rudin (1991, p.78). If $f : K \to \ell^\infty$ is continuous and the closure of $cof(K)$ is compact in $\ell^\infty$, then $f$ is Pettis integrable, where $coS$ is the convex hull of a subset $S$ of a vector space.

An economy is then a probability measure $\mu$ on the measurable space $(\mathcal{P} \times \Omega, \mathcal{B}(\mathcal{P} \times \Omega))$. We will denote the economy under consideration by $\mu$. The marginals of $\mu$ will be denoted by subscripts, for instance, the marginal on $\mathcal{P}$ is $\mu_{\mathcal{P}}$ and so on. The similar definitions on a distribution on $X \times \mathcal{P} \times \Omega$, see the next Definition.

Definition. A pair $(\boldsymbol{p}, \nu)$ of a price vector $\boldsymbol{p} \in \ell^1_+$ and a probability measure $\nu$ on $X \times \mathcal{P} \times \Omega$ is called a competitive equilibrium of the economy $\mu$ if the following conditions hold.

(E-1) $\nu_{\mathcal{P} \times \Omega} = \mu$,

(E-2) $\nu(\{(\boldsymbol{x}, \precsim, \omega) \in X \times \mathcal{P} \times \Omega |\ \boldsymbol{px} \le \boldsymbol{p}\omega \text{ and } \boldsymbol{x} \succsim \boldsymbol{w} \text{ whenever } \boldsymbol{pw} \le \boldsymbol{p}\omega\}) = 1$,

(E-3) $\int_X \boldsymbol{x} d\nu_X \le \int_\Omega \omega d\mu_\Omega$.

The condition (E-1) says that the distribution of agents' characteristics induced by $\nu$ coincides with the economy $\mu$, and the condition (E-2) says that almost all consumers maximize their utilities under their budget constraints. For $\boldsymbol{x} \in X = P \times Z$, let $\boldsymbol{x} = (\boldsymbol{y}, \boldsymbol{z})$, $\boldsymbol{y} \in P$ and $\boldsymbol{z} \in Z$. Correspondingly, we denote $\omega = (\omega^t) = (\omega_\ell, \omega_\infty)$, $\omega_\ell = (\omega^1 \dots \omega^\ell)$, $\omega_\infty = (\omega^{\ell+1} \dots)$. In the condition (E-3) which says that the total demand is equal to the total endowment (the market condition), $\int_\Omega \omega d\mu_\Omega$ exists by virtue of Fact 4 and it is equivalent to

(E-3') $\int_P \boldsymbol{y} d\nu_P \le \int_C \omega_\ell d\mu_C$, and $\int_Z \boldsymbol{z} d\nu_Z \le \int_\Gamma \omega_\infty d\mu_\Gamma$, where we set $\Omega = C\times\Gamma$, $C = \{\omega \in \mathbb{R}^\ell |\ 0 \leq \omega^t \leq \gamma,\ t = 1 \dots \ell\}$ and $\Gamma = \{\omega \in \ell^\infty |\ 0 \leq \omega^t \leq \gamma,\ t \geq 1\}$.

For vector $\boldsymbol{x} = (x^t) \in \ell^\infty$, we denote by $\boldsymbol{x} \ggg \mathbf{0}$ if and only if there exists an $\epsilon > 0$ such that $x^t \geq \epsilon$ for all $t \geq 1$. The following assumptions will be used in order to male every consumer's income to be positive.

Assumption (E). $\int_\Omega \omega d\mu_\Omega \ggg \mathbf{0}$,

Assumption (P). $\mu_\Omega(\{\omega = (\omega_\ell, \omega_\infty) \in \Omega |\ \omega_\ell > \mathbf{0}\}) = 1$.

Let $I = [0, 1]$ be the unit interval on $\mathbb{R}$ and $\lambda$ be the Lebesgue measure on $I$. A measurable map $\mathcal{E} : I \to \mathcal{P}\times\Omega$ such that $\mu = \lambda\circ\mathcal{E}^{-1}$ is called a representation of the economy $\mu$. If $\mathcal{P} \times \Omega$ is a compact metric space, and indeed we have assumed to be so, the representation of $\mu$ exists by the Skorokhod's theorem,

Fact 5 (Hildenbrand (1974, p.50)). Let $K$ be a complete and separable metric space and $(\xi_n)$ a weakly converging sequence of measures on $K$ with the limit $\xi$. Then there exist measurable mappings $f$ and $f_n (n \in \mathbb{N})$ on the unit interval $I = [0, 1]$ to $K$ such that $\xi = \lambda\circ f^{-1}$, $\xi_n = \lambda\circ f_n^{-1}$, and $f_n \to f$ a.e. in $I$, where $\lambda$ is the Lebesgue measure on $I$.

Let $\mathcal{E} : I \to \mathcal{P}\times\Omega$ be the representation of $\mu$. Note that there need not be the case that there exists a map $f : I \to X$ such that $\nu = \lambda \circ (f, \mathcal{E})^{-1}$. If such an $f$ existed, it could be naturally called an assignment. However, it will be true that such an $f$ does exist for some representation. This point is primarily technical, and interested readers can consult to Mas-Colell (1975).

The main result of this paper now reads

Theorem 1. Let $\mu$ be an economy which satisfies the assumptions (E) and (P). Then there exists a competitive equilibrium $(\boldsymbol{p}, \nu)$ for $\mu$.

Proof. Since $\mathcal{P}\times\Omega$ is a compact metric space, by Skorokhod's theorem (Fact 5), there exists a measurable map

$$\mathcal{E} : I \to \mathcal{P}\times\Omega, a \mapsto \mathcal{E}(a) = (\succsim_a, \omega(a)),$$

such that $\mu = \lambda \circ \mathcal{E}^{-1}$, where $I = [0, 1]$ and $\lambda$ is the Lebesgue measure on $[0, 1]$.

For each $n \in \mathbb{N}$, set $K^n = \{\boldsymbol{x} = (x^t) \in \ell^\infty | \boldsymbol{x} = (x^1, x^2 \dots x^n, 0, 0 \dots)\}$. Naturally we can identify $K^n$ with $\mathbb{R}^n$, or $K^n \approx \mathbb{R}^n$. Define

$$X^n = X \cap K^n,\ \succsim^n = \succsim \cap (X^n \times X^n), \mathcal{P}^n = \mathcal{P} \cap \mathcal{F}(X^n \times X^n),\ \Omega^n = \Omega \cap K^n,$$

and for every $\omega = (\omega^t) \in \Omega$, we denote $\omega_n = (\omega^1, \omega^2 \dots \omega^n, 0, 0 \dots) \in \Omega^n$. Then they induce finite dimensional representations $\mathcal{E}^n : I \to \mathcal{P}^n \times \Omega^n$ defined by $\mathcal{E}^n(a) = (\succsim_a^n, \omega_n(a))$, $n = 1, 2 \dots$. By Theorem A1 (Appendix), for each $n$, there exist a price vector $\boldsymbol{p}_n \in \mathbb{R}_+^n$ and an allocation $(\boldsymbol{x}_n(a))$ which satisfy

$$\boldsymbol{p}_n \boldsymbol{x}_n(a) \leq \boldsymbol{p}_n \omega_n(a) \text{ and } \boldsymbol{x}_n(a) \succsim \boldsymbol{w} \text{ whenever } \boldsymbol{p}_n \boldsymbol{w} \leq \boldsymbol{p}_n \omega_n(a) \text{ a.e,} \tag{1}$$

and

$$\int_I \boldsymbol{x}_n(a)da \le \int_I \omega_n(a)da. \tag{2}$$

Without loss of generality, we can assume that $\boldsymbol{p}_n\mathbf{1} = \sum_{t=1}^n p_n^t = 1$ for all $n$, where $\boldsymbol{p}_n = (p_n^t)$ and $\mathbf{1} = (1, 1 \dots)$. Writing $\boldsymbol{x}_n(a) = (\boldsymbol{y}_n(a), \boldsymbol{z}_n(a)) \in P \times Z^n$, where $Z^n = Z \times K^n$, we define $\nu_P^n = \lambda \circ \boldsymbol{y}_n^{-1}$, $\nu_{Z\times\mathcal{P}\times\Omega}^n = \lambda \circ (\boldsymbol{z}_n, \mathcal{E}^n)^{-1}$ and $\nu^n = \nu_P^n \times \nu_{Z\times\mathcal{P}\times\Omega}^n$. Note that $\nu^n$ is the product measure $\nu_P^n$ and $\nu_{Z\times\mathcal{P}\times\Omega}^n$ which is a probability measure on $X \times \mathcal{P}\times\Omega$, since $X^n \times \mathcal{P}^n \times \Omega^n \subset X \times \mathcal{P}\times\Omega$ for each $n$. Then we have that

$$\nu^n(\{(\boldsymbol{x}_n, \succsim^n, \omega_n) \in X \times \mathcal{P}\times\Omega |\ \boldsymbol{p}_n\boldsymbol{x}_n \le \boldsymbol{p}_n\omega_n \text{ and } \boldsymbol{x}_n \succsim^n \boldsymbol{w} \text{ whenever } \boldsymbol{p}_n\boldsymbol{w} \le \boldsymbol{p}_n\omega_n\}) = 1, \tag{3}$$

and

$$support(\nu^n) \subset X^n \times \mathcal{P}^n \times \Omega^n, \ \ n = 1, 2 \dots, \tag{4}$$

and it follows from (3) that for each $n$ and for every $\boldsymbol{y}_n \in support(\nu_P^n)$,

$$\nu_{Z\times\mathcal{P}\times\Omega}^n(\{(\boldsymbol{z}_n, \succsim^n, \omega_n) \in X \times \mathcal{P}\times\Omega |\ \boldsymbol{p}_n(\boldsymbol{y}_n, \boldsymbol{z}_n) \le \boldsymbol{p}_n\omega_n \text{ and } (\boldsymbol{y}_n, \boldsymbol{z}_n) \succsim^n \boldsymbol{w} \text{ whenever } \boldsymbol{p}_n\boldsymbol{w} \le \boldsymbol{p}_n\omega_n\}) = 1. \tag{5}$$

Since the space of probability measures on a compact metric space is a compact metric space (Hildenbrand (1974, p.49)), there exists a probability measure $\nu_{Z\times\mathcal{P}\times\Omega}$ such that $\nu_{Z\times\mathcal{P}\times\Omega}^n \to \nu_{Z\times\mathcal{P}\times\Omega}$ in the weak* topology of probability measures.

On the other hand, by the Fatou's lemma in $\ell$-dimension (Hildenbrand (1974, p.69)), we have an integrable function $\boldsymbol{y} : I \to \mathbb{R}^\ell$ such that $\boldsymbol{y}_n(a) \to \boldsymbol{y}(a)$ a.e. and that $\int_I \boldsymbol{y}(a)da \le \int_I \omega_\ell da$. Then it follows Fact 6 below that $\nu_P^n = \lambda \circ \boldsymbol{y}_n^{-1} \to \nu_P = \lambda \circ \boldsymbol{y}^{-1}$.

Fact 6 (Hildenbrand (1974, (2) and (37))). Let $(A, \mathcal{A}, \xi)$ be a probability space and $(S, \rho)$ be a separable metric space. If $f_n, f$ are measurable functions from $A$ to $S$ and $\rho(f_n(a), f(a)) \to 0$ a.e. in $\xi$, then $\xi \circ f_n^{-1} \to \xi \circ f^{-1}$.

Consequently we have $\nu_P^n \times \nu_{Z\times\mathcal{P}\times\Omega}^n \to \nu_P \times \nu_{Z\times\mathcal{P}\times\Omega} \equiv \nu$ by applying

Fact 7 (Hildenbrand (1974, (27))). Let $(\mu_n)$ and $\nu_n)$ be sequences of measures on the separable metric spaces $S$ and $T$, respectively. Then the sequence $(\mu_n \times \nu_n)$ of product measures on $S\times T$ converges weakly to the product measure $\mu \times \nu$ on $S \times T$ if and only if $(\mu_n)$ converges weakly to $\mu$ and $(\nu_n)$ converges weakly to $\nu$.

Since the set $\Delta = \{\pi \in ba_+ |\ \|\pi\| = \pi\mathbf{1} = 1\}$ is weak* compact by the Alaoglu's theorem (Fact 3), we have a price vector $\pi \in ba_+$ with $\pi\mathbf{1} = 1$ and such that $\boldsymbol{p}_n \to \pi$ in the $\sigma(ba, \ell^\infty)$-topology. We can write $\pi = (\pi_\ell, \pi_\infty)$, $\pi_\ell \in \mathbb{R}^\ell$ and $\pi_\infty \in ba$.

We show that $X^n \times X^n \to X \times X$ in the topology of closed convergence $\tau_c$. It is clear that $Li(X^n \times X^n) \subset Ls(X^n \times X^n) \subset X \times X$. Therefore it su ces to show that $X \times X \subset Li(X^n \times X^n)$. Let $(\boldsymbol{x}, \boldsymbol{w}) = ((x^t), (w^t)) \in X \times X$, and set $\boldsymbol{x}_n = (x^1 \ldots x^n, 0, 0 \ldots)$ and similarly $\boldsymbol{w}_n$ for $\boldsymbol{w}$. Then $(\boldsymbol{x}_n, \boldsymbol{w}_n) \in X^n \times X^n$ for all $n$ and $(\boldsymbol{x}_n, \boldsymbol{w}_n) \to (\boldsymbol{x}, \boldsymbol{w})$. Hence $(\boldsymbol{x}, \boldsymbol{w}) \in Li(X^n \times X^n)$. Then it follows that $\succsim^n = \succsim \cap (X^n \times X^n) \to \succsim$. Obviously one obtains $\omega_n \to \omega$ in the $\sigma(\ell^\infty, \ell^1)$-topology. Consequently we have $\mathcal{E}^n(a) \to \mathcal{E}^n(a)$ a.e. on $I$. It follows from Fact 6 that $\nu^n_{\mathcal{P}\times\Omega} = \lambda \circ (\mathcal{E}^n)^{-1} \to \lambda \circ \mathcal{E}^{-1} = \mu$, or $\nu_{\mathcal{P}\times\Omega} = \mu$.

Furthermore, since $\int_I \boldsymbol{x}_n(a)da = \int_X \boldsymbol{x} d\nu^n_X \leq \int_\Omega \omega d\nu^n_\Omega = \int_I \omega_n(a)da$ and $\nu^n_\Omega \to \mu_\Omega$, $\nu^n_X \to \nu_X$, we have

$$\int_X \boldsymbol{x} d\nu_X = \lim_{n\to\infty} \int_X \boldsymbol{x} d\nu^n_X \leq \lim_{n\to\infty} \int_\Omega \omega d\nu^n_\Omega = \int_\Omega \omega d\nu_\Omega = \int_\Omega \omega d\mu_\Omega,$$

in which the first and the second equality follow from

Lemma 1. Let $f : X \to \ell^\infty$ be weak* continuous and let $\nu^n, \nu$ be probability measures on $X$ with $\nu^n \to \nu$. Then it follows that $\int_X f(\boldsymbol{x})d\nu^n \to \int_X f(\boldsymbol{x})d\nu$ in the weak* topology.

Proof. Let $\boldsymbol{q} \in \ell^1$. Then $\boldsymbol{q}f(\boldsymbol{x})$ is a continuous function on $X$. Since $\nu^n \to \nu$ in the weak* topology of probability measures, we have

$$\boldsymbol{q}\int_X f(\boldsymbol{x})d\nu^n = \int_X \boldsymbol{q}f(\boldsymbol{x})d\nu^n \to \int_X \boldsymbol{q}f(\boldsymbol{x})d\nu = \boldsymbol{q}\int_X f(\boldsymbol{x})d\nu,$$

hence $\int_X f(\boldsymbol{x})d\nu^n \to \int_X f(\boldsymbol{x})d\nu$ in the $\sigma(\ell^\infty, \ell^1)$-topology. ∎

Finally, we need only to show that $\nu(E) = 1$, where

$$E = \{(\boldsymbol{x}, \succsim, \omega) \in X \times \mathcal{P}\times\Omega |\ \boldsymbol{p}\boldsymbol{x} \leq \boldsymbol{p}\omega \text{ and } \boldsymbol{x} \succsim \boldsymbol{w} \text{ whenever } \boldsymbol{p}\boldsymbol{w} \leq \boldsymbol{p}\omega\}.$$

In order to prove this, we need

Fact 8 (Mas-Colell (1975)). Let $K$ be a compact metric space. If $F_n$ is a sequence of closed subsets of $K$ such that $F_n \to F$ in the topology of closed convergence and $\mu_n$ is a sequence of probability measures on $K$ such that $\mu_n(F_n) = 1$ for all $n$ and $\mu_n \to \mu$, then $\mu(F) = 1$.

Note that it follows from Fact 8 that if $F_n$ is a sequence of closed subsets of a compact metric space $K$ and $\mu_n$ is a sequence of probability measures on $K$ such that $\mu_n(F_n) = 1$ for all $n$ and $\mu_n \to \mu$, then $\mu(Ls(F_n)) = 1$. Indeed, since the space $\mathcal{F}(K)$ of all closed subsets of $K$ is a compact metric space, we can extract a converging subsequence $(F_{n_i})$ of $F_n$ with $F_{n_i} \to F \subset K$. Then by Fact 8, $1 = \mu(F) = \mu(Ls(F_{n_i})) \leq \mu(Ls(F_n)) \leq 1$.

Define

$$F = \{(\boldsymbol{x}, \succsim, \omega) \in X \times \mathcal{P}\times\Omega |\ \pi\boldsymbol{w} \leq \pi\omega \text{ implies that } \boldsymbol{x} \succsim \boldsymbol{w}\}.$$

We shall shoe that $\nu(F)=1$. In order to prove this, we define for each $\boldsymbol{y}\in support(\nu_P)$,

$$F(\boldsymbol{y})=\{(\boldsymbol{z},\succsim,\omega)\in Z\times\mathcal{P}\times\Omega|\ \ \pi\boldsymbol{w}\leq\pi\omega\ \text{ implies that }\ (\boldsymbol{y},\boldsymbol{z})\succsim\boldsymbol{w}\}$$

and we will first prove that

$$\nu_{Z\times\mathcal{P}\times\Omega}(F(\boldsymbol{y}))=1\ \text{ for every }\ \boldsymbol{y}\in support(\nu_P).$$

Then by the Fubini's theorem (1974, (24)), it follows that

$$\nu(F)=\int_{P\times Z\times\mathcal{P}\times\Omega}\chi_F d\nu=\int_P\nu_P\int_{Z\times\mathcal{P}\times\Omega}\chi_{F(\boldsymbol{y})}d\nu_{Z\times\mathcal{P}\times\Omega}=\nu_P(P)=1,$$

where $\chi_A$ is the characteristic function od a set $A$.

Let $\boldsymbol{y}\in support(\nu_P)$. Then we can take a sequence $\{\boldsymbol{y}_n\}$ such that $\boldsymbol{y}_n\to\boldsymbol{y}$ and $\boldsymbol{y}_n\in support(\nu_P^n)$ for all $n$. Define

$$F^n(\boldsymbol{y}_n)=\{(\boldsymbol{z}_n,\succsim^n,\omega_n)\in Z^n\times\mathcal{P}^n\times\Omega^n|\ \ \boldsymbol{p}_n\boldsymbol{w}\leq\boldsymbol{p}_n\omega_n\ \text{ implies that }\ (\boldsymbol{y}_n,\boldsymbol{z}_n)\succsim^n\boldsymbol{w}\}.$$

Then by (5), we have $\nu_{Z\times\mathcal{P}\times\Omega}(F^n(\boldsymbol{y}_n))=1$ for all $n$. Since $\nu^n_{Z\times\mathcal{P}\times\Omega}\to\nu_{Z\times\mathcal{P}\times\Omega}$, it su ces from Fact 8 to prove that $Ls(F^n(\boldsymbol{y}_n))\subset F(\boldsymbol{y})$.

To prove this, take a sequence $(\boldsymbol{z}_n,\succsim^n,\omega_n)\in F^n(\boldsymbol{y}_n)$ with $(\boldsymbol{z}_n,\succsim^n,\omega_n)\to(\boldsymbol{z},\succsim,\omega)\in Z\times\mathcal{P}\times\Omega$. We have to show that $(\boldsymbol{z},\succsim,\omega)\in F(\boldsymbol{y})$. Suppose not. Then there exists $\boldsymbol{w}=(w^t)\in X$ such that $\pi\boldsymbol{w}\leq\pi\omega$ and $(\boldsymbol{y},\boldsymbol{z})\prec\boldsymbol{w}$. First we shall show that $\pi_\ell\gg\mathbf{0}$. Suppose not. Then we have $p^t_n\to\pi^t=0$ for some $t=1\ldots\ell$. Since $\pi\int_\Omega\omega\mu_\Omega>0$, it follows that $\pi(\{s\})>0$ and $\pi(\{s\})\omega^s(a)>0$ for some $s\neq t$ on a set with positive measure. Let $\pi(\{s\})\omega^s(a)>0$ and take $\epsilon>0$ with $\omega^s(a)-\epsilon\geq0$. For each $n$, take $\delta_n>0$ so as to $p^t_n\delta_n-p^s_n\epsilon=0$. Defining $\boldsymbol{w}_n=\omega(a)+\delta_n\boldsymbol{e}_t-\epsilon\boldsymbol{e}_s$, we have $\boldsymbol{p}_n\boldsymbol{w}_n=\boldsymbol{p}_n\omega$ and $\delta_n=(p^s_n/p^t_n)\epsilon\to\infty$. Hence $\boldsymbol{x}_n(a)\prec^n\boldsymbol{w}_n$ for $n$ su ciently large, a contradiction. Indeed, since the set $\{\boldsymbol{x}_n(a)|\ n\in\mathbb{N}\}$ is contained in a compact subset $C$ of $X$, for each $\boldsymbol{x}\in C$, one obtains $\delta_{\boldsymbol{x}}>0$ such that $\boldsymbol{x}\prec\omega(a)+\delta_{\boldsymbol{x}}\boldsymbol{e}_t-\epsilon\boldsymbol{e}_s$. Since $\prec$ is continuous, there exists a neighborhood $U_{\boldsymbol{x}}$ of $\boldsymbol{x}$ such that $\boldsymbol{z}\prec\omega(a)+\delta_{\boldsymbol{x}}\boldsymbol{e}_t-\epsilon\boldsymbol{e}_s$ for every $\boldsymbol{z}\in U_{\boldsymbol{x}}$. Since $C$ is compact, we can take $\boldsymbol{x}_1\ldots\boldsymbol{x}_N$ such that $C\subset\cup_{i=1}^N U_{\boldsymbol{x}_i}$. Take an $n$ such that $\delta_n>max\{\delta_{\boldsymbol{x}_1}\ldots\delta_{\boldsymbol{x}_N}\}$. Then $\boldsymbol{x}_n(a)\prec^n\boldsymbol{w}_n$. Since $\nu_\Omega(\{\omega\in\Omega|\ \omega^t>0\ \text{for some}\ t=1\ldots\ell\})=1$ and $\pi_\ell\gg\mathbf{0}$, it follows that $\nu_\Omega(\{\omega\in\Omega|\ \pi\omega>0\})=1$, hence we can assume that $\pi\omega>0$.

Since the preferences are continuous, we can assume without loss of generality that $\pi\boldsymbol{w}<\pi\omega$ and $(\boldsymbol{y},\boldsymbol{z})\prec\boldsymbol{w}$. Let $\boldsymbol{w}_n=(w^1\ldots w^n,0,0\ldots)$ be the projection of $\boldsymbol{w}$ to $K^n$. Since $\boldsymbol{w}_n\to\boldsymbol{w}$ in the $\sigma(\ell^\infty,\ell^1)$-topology, we have for su ciently large $N$ that $\pi\boldsymbol{w}_N\leq\pi\boldsymbol{w}<\pi\omega$ and $\boldsymbol{x}\prec\boldsymbol{w}_N$, since $\pi\geq\mathbf{0}$ and $\boldsymbol{w}_N\leq\boldsymbol{w}$. Since $\boldsymbol{p}_n\to\pi$ and $(\boldsymbol{y}_n,\boldsymbol{z}_n)\to(\boldsymbol{y},\boldsymbol{z})$, it follows that for some $n\geq N$, $0\leq\boldsymbol{p}_n\boldsymbol{w}_N<\boldsymbol{p}_n\omega=\boldsymbol{p}_n\omega_n$ and $(\boldsymbol{y}_n,\boldsymbol{z}_n)\prec\boldsymbol{w}_N$, or $\boldsymbol{x}_n\prec^n\boldsymbol{w}_N$. This contradicts the fact that $(\boldsymbol{z}_n,\succsim^n,\omega_n)\in F^n(\boldsymbol{y}_n)$.

We have then obtained

$$\nu(\{(\boldsymbol{x},\succsim,\omega)\in X\times\mathcal{P}\times\Omega|\ \ \boldsymbol{x}\prec\boldsymbol{w}\ \text{ implies that }\ \pi\omega<\pi\boldsymbol{w}\})=1.$$

Let $\boldsymbol{x} \prec \boldsymbol{w}$. Then we can assume that $\boldsymbol{x} \prec \boldsymbol{w}_n$ and $\pi \boldsymbol{w}_n > \pi\omega$ for $n$ su ciently large. By the Yosida-Hewitt's theorem (Fact 1), we can write $\pi = \pi_c + \pi_p$, and we denote $\pi_c = \boldsymbol{p}$. We will show that $(\boldsymbol{p}, \nu)$ is an equilibrium of the economy $\mu$. Since $\pi_p$ is purely finitely additive, $\pi_p(\{1 \dots n\}) = 0$ for each $n$. It follows from this and $\pi_c \geq \boldsymbol{0}$ that

$$\pi \boldsymbol{w}_n = (\pi_c + \pi_p)\boldsymbol{w}_n = \pi_c \boldsymbol{w}_n \leq \pi_c \boldsymbol{w} = \boldsymbol{p}\boldsymbol{w},$$

since $\boldsymbol{w}_n \leq \boldsymbol{w}$. On the other hand, $\pi_p \geq \boldsymbol{0}$ and $\omega \geq \boldsymbol{0}$ imply that $\pi\omega = (\pi_c + \pi_p)\omega \geq \pi_c\omega = \boldsymbol{p}\omega$, and consequently we have $\boldsymbol{p}\boldsymbol{w} > \boldsymbol{p}\omega$. Summing up, we have verified that

$$\nu(\{(\boldsymbol{x}, \succsim, \omega) \in X \times \mathcal{P} \times \Omega | \ \boldsymbol{x} \prec \boldsymbol{w} \text{ implies that } \boldsymbol{p}\omega < \boldsymbol{p}\boldsymbol{w}\}) = 1. \tag{6}$$

Since the preferences are locally non-satiated, there exists $\boldsymbol{w} \in X$ arbitrarily close to $\boldsymbol{x}$ such that $\boldsymbol{x} \prec \boldsymbol{w}$, therefore we have

$$\nu_{X \times \Omega}((\boldsymbol{x}, \omega) \in X \times \Omega | \ \boldsymbol{p}\boldsymbol{x} \geq \boldsymbol{p}\omega\}) = 1. \tag{7}$$

On the other hand, we have shown that

$$\int_X \boldsymbol{x} d\nu_X \leq \int_\Omega \omega d\nu_\Omega,$$

so that

$$\int_X \boldsymbol{p}\boldsymbol{x} d\nu_X = \boldsymbol{p} \int_X \boldsymbol{x} d\nu_X \leq \boldsymbol{p} \int_\Omega \omega d\nu_\Omega = \int_\Omega \boldsymbol{p}\omega d\nu_\Omega. \tag{8}$$

Combining (7) and (8), we have

$$\nu_{X \times \Omega}(\{(\boldsymbol{x}, \omega) \in X \times \Omega | \ \boldsymbol{p}\boldsymbol{x} = \boldsymbol{p}\omega\}) = 1. \tag{9}$$

From (6) and (9), we obtain $\nu(E) = 1$, and the proof of the theorem is complete. ∎

## 3 The Case of Indivisible Commodities

In this section, we discuss a model of large exchange economies containing the commodities which are consumed in the integer unit, or the indivisible commodities.

We still assume that the primary commodities are divisible, hence $x^t \in \mathbb{R}$ for $t = 1 \dots \ell$, but the commodities $t \geq \ell + 1$ are indivisible, so that $x^t \in \mathbb{N}$ for $t \geq \ell + 1$. The consumption set $X_{idv}$ of each consumer in this section is then given by

$$X_{idv} = \{\boldsymbol{x} = (x^t) \in \ell^\infty | \ 0 \leq x^t < +\infty \text{ for } 1 \leq t \leq \ell, x^t \in \mathbb{N}, x^t \leq \beta \text{ for } t \geq \ell + 1\}.$$

As in the previous section, $X_{idv}$ is isomorphic to $\mathbb{R}^\ell_+ \times Z_{idv} = P \times Z_{idv}$, where $Z_{idv} = \{\boldsymbol{x} = (x^t) \in \ell^\infty_+ | \ x^t \in \mathbb{N}, x^t \leq \beta \text{ for all } t\}$. Similarly we define the set of allowed endowments $\Omega_{idv}$ by

$$\Omega_{idv} = \{\omega = (\omega^t) | \ 0 \leq \omega^t \leq \gamma \text{ for all } t, \omega^t \in \mathbb{N} \text{ for } t \geq \ell + 1\}$$

and we sometimes write $\Omega_{idv} = \Omega_\ell \times \Omega_\infty \subset \mathbb{R}^\ell_+ \times Z_{idv}$.

The space of allowed preferences $\mathcal{P}_{idv} \subset \mathcal{F}(X_{idv} \times X_{idv})$ is defined as a compact set of continuous preorders which satisfy the conditions $(i)$,$(ii)$ and $(iii)$ of the previous section. An economy $\mu$ is a probability measure on $(\mathcal{P}_{idv} \times \Omega_{idv}, \mathcal{B}(\mathcal{P}_{idv} \times \Omega_{idv}))$. The competitive equilibrium $(\boldsymbol{p}, \nu)$ for the economy $\mu$ is defined as in the same way as that of the previous section. In order to handle the indivisible commodities on the infinite dimensional commodity space, we need the extra condition. First we require the primary commodities are suitably "di used" or "dispersed".

Assumption (D). $\mu_{\Omega_\ell}(\{\omega_\ell \in \Omega_\ell | \ \sum_{t=1}^{\ell} p^t \omega^t = w\}) = 0$ for every $\boldsymbol{p} = (p^t) \in \mathbb{R}^\ell_+$ with $\boldsymbol{p} \neq \boldsymbol{0}$ and every $w \in \mathbb{R}$.

On the other hand, we need that the variations of the indivisible endowments of the economies are very small.

Assumption (F). $support(\mu_{\Omega_\infty})$ is a finite set.

We now state an equilibrium existence theorem with indivisible commodities.

Theorem 2. Let $\mu$ be an economy on $(\mathcal{P}_{idv} \times \Omega_{idv}, \mathcal{B}(\mathcal{P}_{idv} \times \Omega_{idv}))$. Suppose that $\mu$ satisfies the assumptions (E), (P), (D) and (F). Then there exists a competitive equilibrium $(\boldsymbol{p}, \nu)$ for $\mu$.

Remark. The assumption (D) implies that the distribution of the primary commodities are "dispersed", namely that the distribution of their market values does not give a positive measure on any specific value for each price system. It was first introduced by Mas-Colell (1977) and generalized in this form by Yamazaki (1978). The role of this condition is well known. By the indivisible commodities, the behavior of the individual demand generally exhibits the discontinuity at some some price vector. However, by virtue of the assumption (D), the mass of the discontinuous consumers will be $0$ at each price vector, hence the aggregate demand preserves the (upper hemi-)continuity. The assumption (F) is really a very strong assumption. A typical economic situation satisfying the assumption (F) in our mind is that the case of the stationary endowment streams, or $support(\Omega_\infty) = \{(\omega_1^t) \dots (\omega_M^t)\}$ with $\omega_i^t = \omega_i \in \mathbb{N}$ for all $t$ with $\omega_i \le \gamma$, $i = 1 \dots M$.

Proof of Theorem 2. The proof is almost the same as that of Theorem 1, so that we shall only give a brief outline. Let $\mathcal{E}^n(a) = (\succsim_a^n, \omega_n(a))$ be the finite dimensional representation in the proof of Theorem 1. The economy $\mathcal{E}^n : I \to \mathcal{P}_{idv}^n \times \Omega_{idv}^n$ satisfies the assumptions of Theorem A1 in Appendix, hence there exists a competitive equilibrium $(\boldsymbol{p}_n, \boldsymbol{x}_n(a))$ for each $n$. Then as in the proof of Theorem 1, we can show that the sequence $(\boldsymbol{p}_n, \nu^n) = (\boldsymbol{p}_n, \lambda \circ (\mathcal{E}^n, \boldsymbol{x}_n)^{-1})$ converges to a pair of a price vector $\pi \in ba$ and a probability measure $\nu$ on $X_{idv} \times \mathcal{P}_{idv} \times \Omega_{idv}$ such that $\nu_{\mathcal{P}_{idv} \times \Omega_{idv}} = \mu$. From now on, we suppress $idv$ for simplicity. Define

$$F = \{(\boldsymbol{x}, \succsim, (\omega_\ell, \omega_\infty)) \in X \times \mathcal{P} \times \Omega | \ \pi \boldsymbol{w} \le \pi\omega \ \text{implies that} \ \boldsymbol{x} \succsim \boldsymbol{w}\}, \tag{10}$$

$$F(\boldsymbol{y}) = \{(\boldsymbol{z}, \succsim, (\omega_\ell, \omega_\infty)) \in Z \times \mathcal{P} \times \Omega | \ \pi \boldsymbol{w} \le \pi\omega \ \text{implies that} \ (\boldsymbol{y}, \boldsymbol{z}) \succsim \boldsymbol{w}\}. \tag{11}$$

As in the proof of Theorem 1, we will show that

$$\nu_{Z \times \mathcal{P} \times \Omega}(F(\boldsymbol{y})) = 1 \ \text{for every} \ \boldsymbol{y} \in support(\nu_P),$$

which implies that $\nu(F) = 1$. Let $\boldsymbol{y} \in support(\nu_P)$. Then we can take a sequence $\{\boldsymbol{y}_n\}$ such that $\boldsymbol{y}_n \to \boldsymbol{y}$ and $\boldsymbol{y}_n \in support(\nu_P^n)$ for all $n$. Define

$$F^n(\boldsymbol{y}_n) = \{(\boldsymbol{z}_n, \succsim^n, \omega_n) \in Z^n \times \mathcal{P}^n \times \Omega^n | \ \boldsymbol{p}_n \boldsymbol{w} \le \boldsymbol{p}_n \omega_n \ \text{implies that} \ (\boldsymbol{y}_n, \boldsymbol{z}_n) \succsim^n \boldsymbol{w}\}.$$

Then by (5), we have $\nu_{Z \times \mathcal{P} \times \Omega}(F^n(\boldsymbol{y}_n)) = 1$ for all $n$. Since $\nu_{Z \times \mathcal{P} \times \Omega}^n \to \nu_{Z \times \mathcal{P} \times \Omega}$, it su ces from Fact 8 to prove that $Ls(F^n(\boldsymbol{y}_n)) \subset F(\boldsymbol{y})$.

To prove this, take a sequence $(\boldsymbol{z}_n, \succsim^n, \omega_n) \in F^n(\boldsymbol{y}_n)$ with $(\boldsymbol{z}_n, \succsim^n, \omega_n) \to (\boldsymbol{z}, \succsim, \omega) \in Z \times \mathcal{P} \times \Omega$. We have to show that $(\boldsymbol{z}, \succsim, \omega) \in F(\boldsymbol{y})$. Suppose not. Then there exists $\boldsymbol{w} = (w^t) = (\boldsymbol{w}_\ell, \boldsymbol{w}_\infty) \in X$ such that $\pi \boldsymbol{w} \le \pi\omega$ and $(\boldsymbol{y}, \boldsymbol{z}) \prec \boldsymbol{w}$. Without loss of generality, we can assume that $\pi \boldsymbol{w}_n \le \pi\omega$ and $(\boldsymbol{y}, \boldsymbol{z}) \prec \boldsymbol{w}_n$ for some $n$, where $\boldsymbol{w}_n = (w^1 \dots w^n, 0, 0 \dots)$. $\pi_\ell \gg \boldsymbol{0}$ and the assumption (a) imply that $\pi\omega > 0$. When $\pi \boldsymbol{w}_n < \pi\omega$, we can show exactly in the same way as in the proof of theorem 1 that $(\boldsymbol{z}, \succsim, \omega) \in F(\boldsymbol{y})$. If $\pi \boldsymbol{w}_n = \pi\omega$, then we can assume that $\boldsymbol{w}_n^t > 0$ for some $t = 1 \dots \ell$. Indeed, it follows from the assumption (D) and $\pi_\ell \gg \boldsymbol{0}$ that

$$\nu_\Omega\left(\left\{\omega \in \Omega \middle| \ \pi(\boldsymbol{0}, \boldsymbol{w}_{n,\infty}) = \pi\omega\right\}\right) = \nu_\Omega\left(\left\{\omega \in \Omega \middle| \ \sum_{t=1}^{\ell} \pi^t \omega_\ell^t = \pi_\infty(\boldsymbol{w}_{n,\infty} - \omega_\infty)\right\}\right) = 0,$$

since the set $\{\pi_\infty(\boldsymbol{w}_{n,\infty} - \omega_\infty)\}$ is countable by the assumption (F) and the fact that the set of the sequence of the form $\boldsymbol{w}^t = (w^t)$ with $w^t(\le \gamma) \in \mathbb{N}$ for all $t$ and $w^t = 0$ after some $t$ is countable. Then we have a sequence $\boldsymbol{w}_1 < \boldsymbol{w}_2 < \dots < \to \boldsymbol{w}$. Since the preferences are continuous, $\pi \boldsymbol{w}_i < \pi\omega$ and $(\boldsymbol{y}, \boldsymbol{z}) \prec \boldsymbol{w}_i$. Then we have again $(\boldsymbol{z}, \succsim, \omega) \in F(\boldsymbol{y})$. Let $\pi = \pi_p + \pi_c = \pi_p + \boldsymbol{p}$ be the Yosida-Hewitt decomposition. Then as in the same way, we can show that

$$\nu(\{(\boldsymbol{x}, \succsim, \omega) \in X \times \mathcal{P} \times \Omega | \ \boldsymbol{p}\boldsymbol{x} \le \boldsymbol{p}\omega \ \text{and} \boldsymbol{w} \succsim \boldsymbol{x} \ \text{whenever} \ \boldsymbol{p}\boldsymbol{w} \le \boldsymbol{p}\omega\}) = 1, \tag{12}$$

and

$$\int_X \boldsymbol{x} d\nu_X \leq \int_\Omega \omega d\mu_\Omega. \tag{13}$$

This completes the proof. ∎

Although the assumption (F) is very strong, we can show that there are "many" economies which satisfy the assumption (F) in the following sense.

Let $\mathscr{E}$ be the set of all economies, or $\mathscr{E} = \{\mu \in \mathcal{M}(\mathcal{P}\times\Omega) | \mu(\mathcal{P}\times\Omega) = 1\}$, where $\mathcal{M}(\mathcal{P}\times\Omega)$ is the set of all nonnegative and countably additive set functions on the measurable space $(\mathcal{P}\times\Omega, \mathcal{B}(\mathcal{P}\times\Omega))$. Hildenbrand (1974) introduced a topology on the space of economies $\mathscr{E}$ in the following way. According to him, a sequence of economies $\{\mu_n\}$ converges to an economy $\mu$ if and only if $\mu_n \to \mu$ in the weak topology of the measures, and $\int_\Omega \omega d\mu_n \to \int_\Omega \omega d\mu$. In other words, he introduces the metric $d$ on the space $\mathscr{E}$ defined by $d(\mu,\nu) = \rho(\mu,\nu) + |\int_\Omega \omega d\mu - \int_\Omega \omega d\nu|$, where $\rho$ is the Prohorov metric on $\mathcal{M}(\mathcal{P}\times\Omega)$ (Hildenbrand (1974, p.49).

Since $\Omega_\infty$ is a compact metric space, it is separable. Hence there exists a dense countable subset $\Omega_*$ of $\Omega_\infty$. From Parthasarathy (1967, p.44) we know that $\mu_\infty$ can be approximated in the weak topology of measures which concentrated on finite subsets of $\Omega_*$, $\mu_F^n \to \mu_\infty$. Setting $\mu^n = \mu_{\mathcal{P}\times\Omega_P} \times \mu_F^n$, it is obvious that $\mu_n \to \mu$ and $\int_\Omega \omega d\mu_n \to \int_\Omega \omega d\mu$.

Let $\mathscr{E}_F$ be the set of economies which satisfy the assumption (F). We have then obtained

Theorem 3. $\mathscr{E}_F$ is a dense subset of $\mathscr{E}$.

## 4 Concluding Remarks

1. Rustichini and Yannelis (1991) proved the existence of competitive equilibrium of an exchange economy which is defined as a measurable map $\mathcal{E} : A \to \mathcal{P}\times\Omega$ as Aumann did without assuming the convexity of preferences. In order to do this, they assumed that "there are many more agents than the commodities" in the economy. To be precise, let $C \subset A$ be a nonnull coalition and let $L_{\infty,C}$ be the set of measurable and essentially bounded functions on $C$. They say that there are many more agents than the commodities if and only if $dimL_{\infty,C} > dim$Commodity Space, where for a vector space $V$, $dimV$ is the cardinality of the (Hamel) basis of $V$. They showed that under this assumption, the Liapunov's theorem hold on an infinite dimensional commodity space which was to be applied to obtain the desired result.

The assumption of Rustichini and Yannelis, however, requires that the space of consumers is at least as "big" as an uncountable product of unit intervals. Therefore the unit interval which was used by Aumann (1966) as a space of consumers will be ruled out as a model for a large number of consumers. As Podczeck (1997) pointed out, one can feel uncomfortable for it, since one wish to interpret an atomless measure space of agents as an idealization of a large but finite number of them.

Podczeck (1997) therefore proposed an alternative assumption. He assumed that "there exist many consumers of every type". Precisely speaking, he considers an equivalence relation between the agents $a$ and $a' \in A$ on the agent's space $(A, \mathcal{A}, \tau)$ defined by $a \sim a'$ if and only if $(\succsim_a, \omega_a) = (\succsim_{a'}, \omega_{a'})$, and calls the quotient space $T = A/\sim$ the space of types. Then his assumption of "many agents of every type $t \in T$" requires that the population measure $\tau$ can be decomposed into a family of measures $\{\tau_t\}_{t\in T}$ such that for (almost) all $t \in T$, the measure $\tau_t$ is atomless and concentrated on the equivalence class $t = \{a | a \in t\} \subset A$.

From the economic point of view, however, his assumption is almost the same as considering the economy as a measure on the space of agents' characteristics in the sense that $T \approx \mathcal{P}\times\Omega$ in a very complicated manner. Consequently, he had to impose some additional assumptions. For example, the $\sigma$-field $\mathcal{A}$ of the agent's space $(A, \mathcal{A}, \tau)$ is assumed to be countably generated. Hence $\mathcal{A}$ itself is countable (Halmos (1974), Theorem C, p.23). Therefore the model of Aumann (1966) or $(A, \mathcal{A}, \tau) = ([0, 1], \mathcal{B}[0, 1], \lambda)$ is excluded.[3]

When at least one of the numbers of agents or commodities is finite, the parametric approach, or the micro-economic approach is economically powerful in the sense that we know everything of each individual in the economy at the equilibrium. However, we are now concerned with a "huge" market in which the both of the agents and the commodities are infinite. When the market scale is very large in the sense that it has a very big population and a large number of commodities are traded, it is usually advisable to see the market from macro-economic view point, and the distribution approach seems to be more natural and appropriate. Indeed, the results of Section 2

[3] When Podczeck applied his general result to the case of the commodity space $L^\infty(X, \mathcal{X}, \sigma)$, he requires that $\mathcal{X}$ is also countably generated. Hence the spaces $\ell^\infty$ and $L^\infty([0, 1], \mathcal{B}[0, 1], \lambda)$ are also ruled out.

and 3 compared to the results presented above seem to justify this opinion.

2. In Theorem 1, the primary commodities are not assumed to be bounded in the consumption set in order to "do jobs" for obtaining the positive income of each individual in the equilibrium. (Also in Theorem 2, they are assumed to be "dispersed" for the regularizing e ect of the aggregate demand.) On the other hand, the non-primary commodities are assumed to be bounded in order to get an equilibrium (marginal) distribution on a (weak*) compact set. Our strategy of the proof is therefore to divide the consumption set to the finite dimensional part and the infinite dimensional part and to use the proofs of the Aumann-Hildenbrand and Mas-Colell-Jones simultaneously for each part. It is an open question to rule out the boundedness assumption on the infinite dimensional part is an open question.

3. It is hard to compare Theorem 2 of section 3 and Theorem 1 of Mas-Colell (1975) both of which contain infinitely many indivisible commodities. In the Mas-Colell's paper, however, the indivisible commodities were not added to the model, but required for the limiting arguments in the proof to go smoothly. This is really an original result on account of the period at which the paper was written, 10 more years before Mas-Colell himself proposed the concept of the proper preferences (see Mas-Colell (1986)) as an alternative condition for the indivisible commodities. Our result certainly did not give any fundamental insight beyond Mas-Colell's result. Rather, it could help our understanding why such a *tour de force* was possible.

# Appendix

Since we did not find elsewhere finite dimensional equilibrium existence theorems with the forms which we need in the text, we will give the proof of them for the completeness. The consumption set of each consumer is $X = \mathbb{R}^\ell_+ \times Z$, where

$$Z = \{\boldsymbol{z} = (z^t) \in \mathbb{R}^m_+ \big|\ z^t \leq \beta^t, t = 1 \dots m\}.$$

A preference relation $\succsim \subset X \times X$ is a complete and transitive binary relation which is closed relative to $X \times X$, satisfies the local nonsatiation and the overriding desirability of the primary commodities. Recall that we denote by $\mathcal{P}$ the set of all allowed preference relations endowed with the topology of closed convergence. Let $(I = [0,1], \mathcal{B}([0,1]), \lambda)$ be the measure space of the unit interval with the Lebesgue measure $\lambda$, and let $\omega : I \to \Omega = \mathbb{R}^\ell_+ \times \mathbb{R}^m$ be an integrable map,

$$\omega : a \mapsto \omega(a), \ \int_I \omega(a) d\lambda < +\infty,$$

which assigns the consumer $a$ his/her endowment vector. An economy $\mathcal{E}$ is a measurable map of $I$ to $\mathcal{P} \times \Omega$,

$$\mathcal{E} : a \mapsto (\succsim_a, \omega(a)) \in \mathcal{P} \times \Omega.$$

A feasible allocation is an integrable map $f$ of $I$ to $X$ such that $\int_I f(a) d\lambda \leq \int_I \omega(a) d\lambda$. A pair of a price vector $\pi \in \mathbb{R}^{\ell+m}_+$ and a feasible allocation $(\pi, f)$ is competitive equilibrium of $\mathcal{E}$ if $\pi f(a) \leq \pi \omega(a)$ and $f(a) \succsim_a \boldsymbol{x}$ whenever $\pi \boldsymbol{x} \leq \pi \omega(a)$ a.e. in $I$.

Theorem A1. An economy $\mathcal{E}$ has an equilibrium if $\int_I \omega(a) \gg \mathbf{0}$ and satisfies

$$\lambda\Big(\{a \in I|\ \omega^t(a) > 0 \ \text{for some} \ t = 1 \dots \ell\}\Big) = 1.$$

Proof. Let $b = \int_I \sum_{t=1}^{\ell+m} \omega^t(a) d\lambda$. Then $b > 0$. For $k = 1, 2, \dots$, define $A_k = \{a \in I|\ \omega(a) \leq kb\mathbf{1}\}$, where $\mathbf{1} = (1 \dots 1) \in \mathbb{R}^{\ell+m}$. Obviously $\emptyset \neq A_1 \subset A_2 \subset \dots$. Let $\mathcal{A}_k = \{C \cap A_k|\ C \in \mathcal{B}([0,1])\}$ and $\lambda_k$ be the restriction of $\lambda$ to $\mathcal{A}_k$. Since $(I, \mathcal{B}([0,1]), \lambda)$ is atomless, so is $(A_k, \mathcal{A}_k, \lambda_k)$ for each $k$. We then define for each $k$ the (truncated) consumption set by $X_k = \{\boldsymbol{x} \in X|\ \boldsymbol{x} \leq kb\mathbf{1}\}$ and the (truncated) demand correspondence

$$\phi_k(a, \pi) = \left\{\boldsymbol{x} \in X_k \Big|\ \pi \boldsymbol{x} \leq \pi \omega(a), \ \text{and if} \ \pi \boldsymbol{x}' \leq \pi \omega(a), \ \text{then} \ \boldsymbol{x} \succsim_a \boldsymbol{x}'\right\},$$

where $\pi \in \Delta = \{\pi = (\pi^t)|\ \sum_{t=1}^{\ell+m} \pi^t = 1\}$. The quasi-demand correspondence is defined by

$$\psi_k(a, \pi) = \begin{cases} \phi_k(a, \pi) & \text{if } \pi \omega(a) > 0 \\ \{\boldsymbol{x} \in X_k|\ \pi \boldsymbol{x} = 0\} & \text{otherwise} \end{cases}$$

Since $X_k$ is compact, it is standard to verify that $\phi_k(a,\pi) \neq \emptyset$, and it follows from Corollary 2 of Hildenbrand (1974, p.104), $\psi_k$ is closed relation, hence it has a measurable graph. We then define the (truncated) mean demand

$$\Phi_k(\pi) = \int_{A_k} \psi_k(a,\pi) d\lambda_k \text{ for } \pi \in \Delta.$$

It is well known that the mean excess demand correspondence $\zeta_k(\pi) = \Phi_k(\pi) - \int_{A_k} \omega(a) d\lambda$ is compact and convex valued (Hildenbrand (1974, p.62)). It is upper hemi-continuous (Hildenbrand (1974, Proposition 8, p.73), satisfies the Walras law: $\pi\zeta_k(\pi) \leq 0$ for every $\pi \in \Delta$.

We can then apply the fixed point theorem (Hildenbrand (1974, p.39))

Lemma A1. Let $C$ be a closed convex cone with the vertex **0** in $\mathbb{R}^{\ell+m}$ which is not a linear subspace. If the correspondence $\zeta$ of $C$ into $\mathbb{R}^{\ell+m}$ is nonempty, compact and convex valued and upper hemi-continuous, and satisfies $\pi\zeta \leq 0$ for every $\pi \in C$, then there exists $\pi^* \in C$ with $\pi^* \neq \mathbf{0}$ such that $\zeta(\pi^*) \in polar(C)$, where $polar(C)$ is the polar of the set $C$,

$$polar(C) = \{\zeta \in \mathbb{R}^{\ell+m} | \ \pi\zeta \leq 0 \text{ for all } \pi \in C\}.$$

Then for each $k$, there exists a price vector $\pi_k \in \Delta$ and an integrable function $f_k(\cdot)$ of $A_k$ to $\mathbb{R}^{\ell+m}$ such that

$$f_k(a) \in \psi_k(a,\pi_k) \text{ a.e. in } A_k \tag{14}$$

$$\int_{A_k} f_k(a) d\lambda_k \leq \int_{A_k} \omega(a) d\lambda_k, \quad k = 1,2\ldots. \tag{15}$$

We extend the domain $A_k$ of $f_k$ to $I$ by defining $f_k(a) = \omega(a)$ for $a \in I \backslash A_k$. Then the condition (14) is replaced by

$$\int_I f_k(a) d\lambda \leq \int_I \omega(a) d\lambda, \quad k = 1,2\ldots. \tag{16}$$

Since $\pi_k \in \Delta$, we can assume that $\pi_k \to \pi \in \Delta$. Since $f_k(a) \in X$ a.e. in $I$ and the set $X$ is bounded from below, it follows from (17) that the sequence $(\int_I f_k(a) d\lambda)$ is bounded. Then by the Fatou's lemma in $\ell$-dimensions (Hildenbrand (1974, p.69)) that there exists an integrable function $f$ of $I$ to $\mathbb{R}^\ell$ such that

$$f(a) \in Ls(f_k(a)) \text{ a.e. in } I, \tag{17}$$

$$\int_I f(a) d\lambda \leq \int_I \omega(a) d\lambda. \tag{18}$$

We complete the proof by showing that

$$f(a) \in \phi(a,\pi) \text{ a.e in } I. \tag{19}$$

It follows from (14) and $f_k(a) \to f(a)$ that $\pi f(a) \le \pi\omega(a)$. Now, there exists a positive integer $k(a)$ for $a$ such that

$$k > k(a) \text{ implies } f_k(a) \in \psi_k(a, \pi_k) \text{ a.e. in } I, \tag{20}$$

for we can take $k(a)$ as a positive integer not smaller than $\|\omega(a)\|/b$. Then $0 \le \omega^t(a)\|\omega(a)\| \le k(a)b$, $t = 1 \ldots \ell + m$. We claim that $\pi^t > 0$ for $t = 1 \ldots \ell$. Suppose not. Then for some $t = 1 \ldots \ell$, $\pi^t = 0$. Since $\pi \int_I \omega(a) d\lambda > 0$ by the assumption (i), $\pi\omega(a) > 0$ on a set $B \subset I$ of positive measure. Then for $a \in B$, $\psi_k(a, \pi_k) = \phi_k(a, \pi_k)$ for $k$ large enough and $\pi^s\omega^s(a) > 0$ for $s \ne t$. Take $\epsilon > 0$ with $\omega^s(a) - \epsilon \ge 0$ and for each $k$, take $\delta_k > 0$ so as to $\pi_k^t \delta_k - \pi_k^s \epsilon = 0$. Define $\boldsymbol{z}_k = \omega(a) + \delta_k \boldsymbol{e}_t - \epsilon \boldsymbol{e}_s$, where $\boldsymbol{e}_t = (0 \ldots 0, 1, 0 \ldots 0)$ and 1 is in the $t$-th coordinate. Then we have $\pi_k \boldsymbol{z}_k = \pi_k \omega(a)$ for all $k$ and $\delta_k = (\pi_k^s/\pi_k^t)\epsilon \to +\infty$, hence $f_k(a) \prec_a \boldsymbol{z}_k$ for $k$ large enough by the overriding desirability of the primary commodities. This contradicts to $f_k(a) \in \phi(a, \pi_k)$. It follows from the assumption (i) that $\pi\omega(a) > 0$ for almost all $a \in I$. For $\boldsymbol{x} \in X$ with $\pi\boldsymbol{x} \le \pi\omega(a)$, we can assume that $\pi\boldsymbol{x} < \pi\omega(a)$. Hence $\pi_k\boldsymbol{x} < \pi_k\omega(a)$ and $f_k(a) \succsim_a \boldsymbol{x}$ for $k$ large enough. By the continuity of $\succsim_a$, we have $f(a) \succsim_a \boldsymbol{x}$. ∎

We now consider the model with indivisible commodities. The consumption set of each consumer is $X_{idv} = \mathbb{R}_+^\ell \times Z$, where

$$Z = \{\boldsymbol{z} = (z^t) \in \mathbb{N}^m \,\big|\, z^t \le \beta^t, t = 1 \ldots m\}.$$

A preference relation $\succsim \subset X_{idv} \times X_{idv}$ is a complete and transitive binary relation which is closed relative to $X_{idv} \times X_{idv}$, satisfies the local nonsatiation and the overriding desirability of the primary commodities. $\mathcal{P}$ is the set of all allowed preference relations endowed with the topology of closed convergence. An economy $\mathcal{E}$ is a measurable map of $I$ to $\mathcal{P} \times \Omega_{idv}$,

$$\mathcal{E} : a \mapsto (\succsim_a, \omega(a)) \in \mathcal{P} \times \Omega_{idv}.$$

A feasible allocation is an integrable map $f$ of $I$ to $X_{idv}$ such that $\int_I f(a) d\lambda \le \int_I \omega(a) d\lambda$. A pair of a price vector $\pi \in \mathbb{R}_+^{\ell+m}$ and a feasible allocation $(\pi, f)$ is competitive equilibrium of $\mathcal{E}$ if $\pi f(a) \le \pi\omega(a)$ and $f(a) \succsim_a \boldsymbol{x}$ whenever $\pi\boldsymbol{x} \le \pi\omega(a)$ a.e. in $I$.

Theorem A2. An economy $\mathcal{E}$ has an equilibrium if $\int_I \omega(a) \gg \mathbf{0}$ and satisfies

(i) $\lambda\Big(\{a \in I \mid \omega^t(a) > 0 \text{ for some } t = 1 \ldots \ell\}\Big) = 1$,

(ii) $\lambda\Big(\{a \in I \mid \sum_{t=1}^{\ell} p^t \omega^t(a) = w\}\Big) = 0$ for every $\boldsymbol{p} = (p^t) \ne \mathbf{0}$ and every $w \in \mathbb{R}$.

Proof. Let $b = \int_I \sum_{t=1}^{\ell+m} \omega^t(a) d\lambda > 0$. For $k = 1, 2, \ldots$, define the measure space $(A_k, \mathcal{A}_k, \lambda_k)$, the consumption set $X_k$ and the (truncated) demand correspondence $\phi_k(a, \pi)$ as in the proof of Theorem A1. Since $X_k$ is compact, it is standard to verify that $\phi_k(a, \pi) \ne \emptyset$, and Proposition 2 of Hildenbrand (1974, p.102) applies also to the case of indivisible commodities, hence $\phi_k$ has a measurable graph. We then define the (truncated) mean demand

$$\Phi_k(\pi) = \int_{A_k} \phi_k(a,\pi) d\lambda_k \ \text{ for } \ \pi \in \Delta.$$

$\Phi(\cdot)$ is compact and convex valued (Hildenbrand (1974, p.62)). In order to show that it is upper hemi-continuous, we restrict its domain. Define

$$Q_k = \mathbb{R}_-^{\ell+m} \cup \{(-\mathbf{1}_\ell, (1/k)\mathbf{1}_m)\}, \ \ k = 1, 2 \ldots,$$

where $\mathbf{1}_\ell = (1 \ldots 1) \in \mathbb{R}^\ell$ and $\mathbf{1}_m = (1 \ldots 1) \in \mathbb{R}^m$, and let $C(Q_k)$ be a closed convex cone with the vertex **0** which is generated by the set $Q_k$. Let $\Delta_k$ be the intersection of its polar set and $\Delta$, or

$$\Delta_k = \{\pi = (\pi^t) \in \mathbb{R}^{\ell+m} | \ \ \pi\zeta \le 0 \ \text{ for each } \ \zeta \in C(Q_k)\} \cap \Delta.$$

Obviously $C(Q_1) \supset C(Q_2) \supset \cdots \supset \mathbb{R}_-^{\ell+m}$ and $C(Q_k) \to \mathbb{R}_-^{\ell+m}$ in the topology of closed convergence, namely that $\mathbb{R}_-^{\ell+m} = Li(C(Q_k)) = Ls(C(Q_k))$. Consequently, $\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta$ and $\Delta = Li(\Delta_k) = Ls(\Delta_k)$. Moreover, if $\pi = (\boldsymbol{p}, \boldsymbol{q}) \in \Delta_k$, where $\boldsymbol{p} \in \mathbb{R}^\ell$ and $\boldsymbol{q} \in \mathbb{R}^m$ with $\pi \neq \mathbf{0}$, then $\boldsymbol{p} \gg \mathbf{0}$ for all $k$. Let $k$ be given. We will show that $\Phi_k : \Delta_k \to X_k$ is upper hemi-continuous. Let $(\pi_n, \xi_n)$ be a sequence in $\Delta_k \times X_k$ such that $(\pi_n, \xi_n) \to (\pi, \xi) \in \Delta_k \times X_k$ and $\xi_n \in \Phi_k(\pi_n)$ for all $n$. Since $X_k$ is compact, it su ces to show that $\xi \in \Phi_k(\pi)$. For each $n$, there exists an integrable function $f_n$ of $A_k$ to $X_k$ with $f_n(a) \in \phi_k(a, \pi_n)$ a.e. in $A_k$ and $\xi_n = \int_{A_k} f_n(a) d\nu$. For the price vector $\pi \in \Delta_k$, define a subset $A_k(\pi)$ of $A_k$ by

$$A_k(\pi) = \{a \in A_k | \ \ \pi(\mathbf{0}, \boldsymbol{z}) = \pi\omega(a) \ \text{ for some } \ \boldsymbol{z} \in \mathbb{N}^m\}.$$

Since $A_k(\pi) = \{a \in A_k | \ \sum_{t=1}^{\ell} \pi^t \omega^t(a) = \sum_{t=1}^{m} \pi^t (z^t - \omega^{\ell+t}(a))\}$ and $z^t, \omega^{\ell+t}(a) \in \mathbb{N}$ for $t = 1 \ldots m$, it follows from the assumption (ii) that $\lambda_k(A_k(\pi)) = 0$, since $\pi^t > 0$, $t = 1 \ldots \ell$. We shall show that

$$Ls(f_n(a)) \subset \phi_k(a, \pi) \ \text{ a.e. in } \ A_k \backslash A_k(\pi). \tag{21}$$

Let $a \in A_k \backslash A_k(\pi)$ be such that $f_n(a) \in \phi_k(a, \pi_n)$ for all $n$, and let $f_n(a) \to f(a) \in X_k$. Since $\pi_n f_n(a) \le \pi_n \omega(a)$ for all $n$, $\pi f(a) \le \pi\omega(a)$. For $\boldsymbol{x} = (x^1 \ldots x^{\ell+m}) \in X_k$ such that $\pi\boldsymbol{x} < \pi\omega(a)$, we have $\pi_n \boldsymbol{x} < \pi_n \omega(a)$ for all $n$ su ciently large. By the continuity of $\succsim_a$, one obtains $f(a) \succsim_a \boldsymbol{x}$. If $\pi f(a) = \pi\omega(a)$, for some $t = 1 \ldots \ell$, there exist $x_i^t \ \ (i = 1, 2, \ldots)$ such that $x_1^t < x_2^t < \cdots \to x^t$, since $a \notin A_k(\pi)$. Let $\boldsymbol{x}_i$ be the vector which is equal to $\boldsymbol{x}$ but whose $t$-th coordinate is replaced by $x_i^t$. Therefore it follows from $\boldsymbol{x}_i \to \boldsymbol{x}$ that $f(a) \succsim_a \boldsymbol{x}$ by the continuity of $\succsim_a$. The claim (21) then implies that

$$\xi \in \int_{A_k} Ls(f_n(a) d\lambda_k \subset \int_{A_k} \phi_k(a, \pi) d\lambda_k = \Phi_k(\pi).$$

We can then apply Lemma A1 (the fixed point theorem) for $\zeta_k(\pi) = \int_{A_k} \phi_k(a,\pi)d\lambda_k - \int_{A_k} \omega(a)d\lambda_k$. Then for each $k$, there exists a price vector $\pi_k \in \Delta_k$ and an integrable function $f_k(\cdot)$ of $A_k$ to $\mathbb{R}^{\ell+m}$ such that

$$f_k(a) \in \phi_k(a,\pi_k) \text{ a.e. in} A_k \tag{22}$$

$$\int_{A_k} f_k(a)d\lambda_k - \int_{A_k} \omega(a)d\lambda_k \in C(Q_k), \;\; k = 1, 2 \dots. \tag{23}$$

We extend the domain $A_k$ of $f_k$ to $I$ by defining $f_k(a) = \omega(a)$ for $a \in I \backslash A_k$. Then the condition (23) is replaced by

$$\int_I f_k(a)d\lambda - \int_I \omega(a)d\lambda \in C(Q_k), \;\; k = 1, 2 \dots. \tag{24}$$

Since $\pi_k \in \Delta_k \subset \Delta$, we can assume that $\pi_k \to \pi \in \Delta$. Since $f_k(a) \in X$ a.e. in $I$ and $X \cap (\int_I \omega(a)d\lambda + C(Q_1))$ is a bounded set, it follows from (24) that the sequence $(\int_I f_k(a)d\lambda)$ is bounded. Then by the Fatou's lemma in $\ell$-dimensions (Hildenbrand (1974, p.69)) that there exists an integrable function $f$ of $I$ to $\mathbb{R}^\ell$ such that

$$f(a) \in Ls(f_k(a)) \text{ a.e. in } I, \tag{25}$$

$$\int_I f(a)d\lambda \leq \int_I \omega(a)d\lambda, \tag{26}$$

since $Ls(C(Q_k)) = \mathbb{R}^{\ell+m}_-$. Now, there exists a positive integer $k(a)$ for $a$ such that

$$k > k(a) \text{ implies } f_k(a) \in \phi_k(a,\pi_k) \text{ a.e. in } I, \tag{27}$$

for we can take $k(a)$ as a positive integer not smaller than $\|\omega(a)\|/b$. Then $0 \leq \omega^t(a)\|\omega(a)\| \leq k(a)b$, $t = 1 \dots \ell + m$.

We complete the proof by showing that

$$f(a) \in \phi(a,\pi) \text{ a.e in } I. \tag{28}$$

It follows from (27) and $f_k(a) \to f(a)$ that $\pi f(a) \leq \pi\omega(a)$. Let $a \in I \backslash A(\pi)$, where $A(\pi) = \{a \in I | \; \pi(\mathbf{0}, \boldsymbol{z}) = \pi\omega(a) \text{ for some } \boldsymbol{z} \in \mathbb{N}^m\}$. For $\boldsymbol{x} \in X$ with $\pi\boldsymbol{x} < \pi\omega(a)$, we have $f(a) \succsim_a \boldsymbol{x}$ as before. If $\pi\boldsymbol{x} = \pi\omega(a)$, there exists a sequence $\boldsymbol{x}_i$ with $\boldsymbol{x}_i \to \boldsymbol{x}$ and $\pi\boldsymbol{x}_i < \pi\omega(a)$, since $a \notin A(\pi)$. Then $f(a) \succsim_a \boldsymbol{x}_i$, hence $f(a) \succsim_a \boldsymbol{x}$ by the continuity of $\succsim_a$. Since $\lambda(A(\pi)) = 0$, the claim (28) is verified and the proof is complete. ∎

## References


Aumann, R.J., (1966) "Existence of Competitive Equilibria in Markets with a Continuum of Traders", *Econometrica* 34, 1-17.

Bewley, T.F., (1970) "Existence of Equilibria with Infinitely Many Commodities," *Journal of Economic Theory* 4, 514-540.

Bewley, T.F., (1991) "A Very Weak Theorem on the Existence of Equilibria in Atomless Economies with Infinitely Many Commodities," in *Equilibrium Theory in Infinite Dimensional Spaces* Ali Khan, M., and N. Yannelis (Eds), Springer-Verlag, Berlin and New York.

Halmos, P., (1974) *Measure Theory*, Graduate Texts in Mathematics, Springer Verlag, Berlin and New York.

Hart,S., W. Hildenbrand, and E. Kohlberg, (1974) "On Equilibrium Allocations as Distributions on the Commodity Space," *Journal of Mathematical Economics* 1, 159-166.

Hildenbrand, W., (1974) *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton, New Jersey.

Jones, L., (1983) "Existence of Equilibria with Infinitely Many Consumers and Infinitely Many Commodities", *Journal of Mathematical Economics* 12, 119-138.

Khan, M., and N.C. Yannelis, (1991) "Equilibria in Markets with a Continuum of agents and Commodities," in *Equilibrium Theory in Infinite Dimensional Spaces* Ali Khan, M., and N. Yannelis (Eds), Springer-Verlag, Berlin and New York.

Mas-Colell, A., (1975) "A Model of Equilibrium with Di erentiated Commodities", *Journal of Mathematical Economics* 2, 263-296.

Mas-Colell, A., (1977) "Indivisible Commodities and General Equilibrium Theory", *Journal of Economic Theory* 16, 443-456.

Mas-Colell, A., (1986) "The Price Equilibrium Existence Problem in a Topological Vector Lattice", *Econometrica* 54, 1039-1054.

Noguchi, M., (1997a) "Economies with a Continuum of Consumers, a Continuum of Suppliers, and an Infinite Dimensional Commodity Space," *Journal of Mathematical Economics* 27, 1-21.

Noguchi, M., (1997b) "Economies with a Continuum of Agents with the Commodity -Price Pairing $(\ell^\infty, \ell^1)$," *Journal of Mathematical Economics* 28, 265-287.

Parthasarathy, K.R., (1967) *Probability Measures on Metric Spaces*, Academic Press, NY.

Podczeck, K., (1997) "Markets with Infinitely Many Commodities and a Continuum of Agents with Non-Convex Preferences," *Economic Theory* 9, 385-426.

Rudin, W., (1991) *Functional Analysis*, McGraw-Hill, NY.

Rustichini, A., and N.C. Yannelis, (1991) "What is Perfect Competition?" in *Equilibrium Theory in Infinite Dimensional Spaces* Ali Khan, M., and N. Yannelis (Eds), Springer-Verlag, Berlin and New York.

Yamazaki., A (1978) "An Equilibrium Existence Theorem without Convexity Assumptions", *Econometrica* 46, 541-555.

Yosida, K., and E. Hewitt, (1956) "Finitely Additive Measures," *Transactions of American Mathematical Society*, 72, 46-66.