Competitive Equilibria of a Large Exchange Economy with Infinite Time Horizon and Indivisible Commodities

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Abstract

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The existence of competitive equilibrium for a large exchange economy over the commodity space ℓ^∞ will be discussed. We define the economy as a measurable map from a measure space to the space of consumers' characteristics following Aumann (1966), and prove the theorem without the convexity of preferences. The case in which the indivisible commodities present will also be discussed.

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1 Introduction

In this paper, we are concerned with an exchange economy with a continuum of consumers introduced by Aumann (1966) and with an infinite time horizon introduced by Bewley (1970 and 1991), respectively. The economy is formulated on the commodity space ℓ^∞,

$$\ell^\infty = \left\{ \xi = (\xi^t) | \ sup_{t\geq 1} |\xi^t| < +\infty \right\},$$

the space of the sequences with bounded supremum norms. We will deal with an exchange economy throughout this paper, hence there exist no producers in the economy.

As we will see in the next section, the space of all summable sequences,

$$\ell^1 = \left\{ \boldsymbol{p} = (p^t) \middle| \ \sum_{t=1}^{\infty} |p^t| < +\infty \right\},$$

is a natural candidate of the price space. The value of a commodity $\xi = (\xi^t) \in \ell^\infty$ evaluated by a price vector $\boldsymbol{p} = (p^t) \in \ell^1$ is then given by the natural "inner product" $\boldsymbol{p}\xi = \sum_{t=1}^{\infty} p^t \xi^t$. Bewley (1970) proved the existence of competitive equilibria for economies with finite number of consumers on this commodity space.

The exchange economy with a measure space $(A, \mathcal{A}, \nu)$ of consumers was first introduced by Aumann (1966) on a finite dimensional commodity space. As is well known, he defined the economy by a measurable map $\mathcal{E} : A \to \mathcal{P} \times \Omega$, where $\mathcal{P}$ is the set of preferences and Ω is the set of endowment vectors. Each element $a \in A$ is interpreted as a "name" of a consumer, and each value of the map $\mathcal{E}(a) = (\succsim_a, \omega_a)$ is the characteristics of the consumer a. He established the existence of the competitive equilibrium of such an economy and observed that the convexity assumption on the preference relations of the consumers is not necessary. This is a consequence of Liapunov's theorem which asserts that on the non-atomic measure space, the integration of a measurable correspondence is a convex set. This means that even if the demand of an individual consumer is not convex valued, the total demand which is defined by the integration over the set A of the individual demand correspondence is convex valued. In the course of the proof of the existence theorem, the ℓ-dimensional version of Fatou's lemma was essentially used.

Several authors have tried to unify the above results of Aumann and Bewley. For example, Khan and Yannelis (1991) and Noguchi (1997a) proved the existence of a competitive equilibrium for the economies with a measure space of agents in which the commodity space is a separable Banach space whose positive orthant has a norm interior point[1]. Bewley (1991) and Noguchi (1997b) proved the equilibrium existence theorems for the economies with a measure space of consumers on the commodity space ℓ^∞. Bewley worked with an exchange economies, and Noguchi (1997a and b) proved his theorems for the economies with continuum of consumers and producers.

[1]Since the space ℓ^∞ is not separable, these results are not considered as generalizations of Bewley (1970).

These authors anticipated on their works that there are significant technical difficulties for extensions of the Aumann's theorem to infinite dimensional commodity spaces. They proved their theorems by approximating the large-infinite dimensional economy by large-finite dimensional sub-economies, a technique which we will also utilize in this paper. In the course of the approximation they were expected to apply the Fatou's lemma as Aumann did. On the finite dimensional spaces, the only condition which is required to the demand correspondences is that they are integrably bounded. The infinite dimensional version of Fatou's lemma, however, requires that the demand correspondences are contained in a convex valued correspondence (see for example, Yannelis (1991) Theorem 5.2). Since the Liapunov convexity theorem fails in the infinite dimensional spaces, this means that the convex valuedness of the demand correspondences themselves is strongly wanted. Indeed, Khan and Yannelis, and Bewley assumed that the preferences are convex. Noguchi assumed that a commodity vector does not belong to the convex hull of its preferred set. These assumptions obviously weaken the impact of the Aumann's classical result which revealed the "convexfying effect" of large numbers of the economic agents.

In this paper, we will prove the existence of competitive equilibria of an exchange economy with a measure space of the consumers on the commodity space ℓ^∞ without any convexity assumptions on the preferences. The key assumption is that the consumptions set X which is identical for all consumers is of the form

$$X = \mathbb{R}^\ell_+ \times Z, \;\; Z = \{\xi = (\xi^t) \in \ell^\infty |\;\; 0 \le \xi^t \le \beta \;\text{ for all }\; t \ge 1\},$$

for some $\beta > 0$, hence the consumption set is the product of the ℓ-dimensional part and the bounded subset of ℓ^∞. We call the first ℓ commodities the primary commodities (Section 2.1).

In Section 3, we will show that for this type of the consumptions set, the finite dimensional approximation works without Fatou's lemma or Liapunov's theorem. Bewley (1991) assumed that $X = \{\xi = (\xi^t) \in \ell^\infty |\; 0 \le \xi^t \le \beta$ for all $t \ge 1\}$ and Noguchi (1997b) also assumed that the consumption set of each consumer is bounded uniformly over consumers. Hence we improved their results essentially, since we discarded the convex preferences, and we relaxed those (uniform) bounded conditions slightly, namely that the primary commodities are not assumed to be bounded in the consumption set[2]

The purpose of including the primary commodities is, other than obtaining less restricted form of the consumption set, to invoke the overriding desirability of the primary commodities (see Section 2.2) and prove that the prices of the primary commodities are positive (Lemma 5 in Section 3). Hence positive amounts of the primary commodities as initial endowments are sufficient for each consumer to have a positive income. Thanks to this, our assumptions on the initial endowments (Assumptions (E) and (P) in Section 2.2) are much weaker than that of the other related literatures. For example, Bewley (1991), Khan-Yannelis (1991), Noguchi (1997a and b), Rustichini-Yannelis

[2] If the commodity space was a general Banach space, the mathematical difficulties for proving existence without the convexity of preferences will be so formidable. For this, see Rustichini-Yannelis (1991) and Podczeck (1997).

(1991) assumed that almost all consumers have his/her initial endowment in the (norm) interior of the consumption set. Obviously such an assumption is very strong in the economies with a continuum of traders.[3]

In Section 4, we will discuss the economy in which the non-primary commodities are consumed only in the integer units, or they prevail in the market as the indivisible commodities along the line of research initiated by Mas-Colell (1977) and Yamazaki (1978). We will prove the existence of competitive equilibria if the endowment assignment for the primary commodities is dispersed in the sense of Mas-Colell and Yamazaki and that of the non-primary commodities is a simple function. Obviously the latter assumption is strong. Fortunately, however, there is a dense subset of the economies satisfying that assumption, hence we can say that there are "many" economies with equilibria.

[3]Precisely, Khan-Yannelis, Noguchi, Rustichini-Yannelis assumed that there exists $\zeta \in X$ such that $\omega - \zeta$ belongs to the (norm) interior of X.

2 The Model and the Result

2.1 Mathematical Preliminaries

As pointed out in Introduction, the commodity space of the economy in this paper is set to be

$$\ell^\infty = \left\{ \xi = (\xi^t) \middle| \ sup_{t\geq 1}|\xi^t| < +\infty \right\},$$

the space of the sequences with bounded supremum norms. It is well known that the space ℓ^∞ is a Banach space with respect to the norm $\|\xi\| = sup_{t\geq 1}|\xi^t|$ for $\xi \in \ell^\infty$ (Royden (1988)).

It is also well known that the dual space of ℓ^∞ is the space of bounded and finitely additive set functions on $\mathbb{N}$ which is denoted by ba,

$$ba = \left\{ \pi : 2^{\mathbb{N}} \to \mathbb{R} \middle| sup_{E\subset\mathbb{N}}|\pi(E)| < +\infty,\ \pi(E \cup F) = \pi(E) + \pi(F) \text{ whenever } E \cap F = \emptyset \right\}.$$

Then we can show that the space ba is a Banach space with the norm

$$\|\pi\| = sup \left\{ \sum_{i=1}^{n} |\pi(E_i)| \middle| \ E_i \cap E_j = \emptyset \text{ for } i \neq j, n \in \mathbb{N} \right\}.$$

Since the commodity vectors are represented by sequences, it is more natural to consider the price vectors also as sequences rather than the set functions. Therefore the subspace ca of ba,

$$ca = \left\{ \pi \in ba \middle| \pi(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \pi(E_n) \text{ whenever } E_i \cap E_j = \emptyset\ (i \neq j) \right\},$$

which is the space of the bounded and countably additive set functions on $\mathbb{N}$ is more appropriate as the price space. Indeed it is easy to see that the space ca is isometrically isomorphic to the space ℓ^1, the space of all summable sequences,

$$\ell^1 = \left\{ \boldsymbol{p} = (p^t) \middle| \ \sum_{t=1}^{\infty} |p^t| < +\infty \right\},$$

which is a Banach space with the norm $\|\boldsymbol{p}\| = \sum_{t=1}^{\infty} |p^t|$.

Then the value of a commodity $\xi = (\xi^t) \in \ell^\infty$ evaluated by a price vector $\boldsymbol{p} = (p^t) \in \ell^1$ is given by the natural "inner product" $\boldsymbol{p}\xi = \sum_{t=1}^{\infty} p^t \xi^t$.

The set function $\pi \in ba$ is called purely finitely additive if $\rho = 0$ whenever $\rho \in ca$ and $0 \leq \rho \leq \pi$. The relation between the ba and ca is made clear by the next fundamental theorem,

Fact 1 (Yosida-Hewitt). If $\pi \in ba$ and $\pi \geq 0$, then there exist set functions $\pi_c \geq 0$ and $\pi_p \geq 0$ in ba such that π_c is countably additive and π_p is purely finitely additive and satisfy $\pi = \pi_c + \pi_p$. This decomposition is unique.

On the space ℓ^∞, we can consider the several topologies. One is of course the norm topology τ_{norm} which was explained above. It is the strongest topology among the topologies which appear in this paper.

The weakest topology in this paper is the product topology τ_d which is induced from the metric

$$d(\xi,\zeta)=\sum_{t=1}^{\infty}\frac{|\xi^t-\zeta^t|}{2^t(1+|\xi^t-\zeta^t|)} \text{ for } \xi=(\xi^t),\zeta=(\zeta^t)\in\ell^\infty.$$

The product topology is nothing but the topology of coordinate-wise convergence, or $\xi=(\xi^t)\to\mathbf{0}$ if and only if $\xi^t\to 0$ for all $t\in\mathbb{N}$.

A net (ξ_α) on ℓ^∞ is said to converge to $\mathbf{0}$ in the weak* topology or $\sigma(\ell^\infty,\ell^1)$-topology if and only if $\boldsymbol{p}\xi_\alpha\to 0$ for each $\boldsymbol{p}\in\ell^1$. The weak* topology is characterized by the weakest topology on ℓ^∞ which makes $(\ell^\infty)^*=\ell^1$, where L^* is the dual space (the set of all continuous linear functionals on L) of a normed linear space L. Then it is stronger than the product topology, since the latter is characterized by $\xi_\alpha\to 0$ if and only if $\boldsymbol{e}_t\xi_\alpha\to 0$ for all for each $\boldsymbol{e}_t=(0\ldots 0,1,0\ldots)\in\ell^1$, where 1 is in the t-th coordinate.

The strongest topology on ℓ^∞ which makes $(\ell^\infty)^*=\ell^1$ is called the Mackey topology $\tau(\ell^\infty,\ell^1)$. It is characterized by saying that a net (ξ_α) on ℓ^∞ is said to converge to $\mathbf{0}$ in $\tau(\ell^\infty,\ell^1)$-topology if and only if $sup\{|\boldsymbol{p}\xi_\alpha||\boldsymbol{p}\in C\}\to 0$ on every $\sigma(\ell^1,\ell^\infty)$-compact, convex and circled subset C of ℓ^1, where a set C is circled if and only if $rC\subset C$ for $-1\le r\le 1$, and the topology $\sigma(\ell^1,\ell^\infty)$ is defined analogously as $\sigma(\ell^\infty,\ell^1)$, namely that a net $(\boldsymbol{p}_\alpha)$ on ℓ^1 is said to converge to $\mathbf{0}$ in the $\sigma(\ell^1,\ell^\infty)$-topology if and only if $\boldsymbol{p}_\alpha\xi\to 0$ for each $\xi\in\ell^\infty$. The topology $\tau(\ell^\infty,\ell^1)$ is weaker than the norm topology. Hence we have $\tau_d\subset\sigma(\ell^\infty,\ell^1)\subset\tau(\ell^\infty,\ell^1)\subset\tau_{norm}$.

Similarly, a net (π_α) on ba is said to converge to $\mathbf{0}$ in the weak* topology or $\sigma(ba,\ell^\infty)$-topology if and only if $\pi_\alpha\xi\to 0$ for each $\xi\in\ell^\infty$.

We can use the next proposition on bounded subsets of ℓ^∞.

Fact 2 (Bewley (1991 a, p.226)). Let Z be a (norm) bounded subset of ℓ^∞. Then on the set Z, the Mackey topology $\tau(\ell^\infty,\ell^1)$ coincides with the product topology τ_d.

In general, let L be a normed vector space and L^* its dual space. A net (ξ_α) in L converges to $\xi\in L$ in the $\sigma(L,L^*)$-topology or weak-topology if and only if $\pi\xi_\alpha\to\pi\xi$ for every $\pi\in L^*$. A net (π_α) in L^* converges to $\pi\in L^*$ in the $\sigma(L^*,L)$-topology or weak*-topology if and only if $\pi_\alpha\xi\to\pi\xi$ for every $\xi\in L$.

Bounded subsets of ℓ^∞ are $\sigma(\ell^\infty,\ell^1)$-weakly compact, namely that the weak* closure of the sets are weak*-compact by the Banach-Alaoglu's theorem.

Fact 3 (Rudin (1991, p.68). If L is a normed space, then the unit ball of L^*, $B=\{\pi\in L^*|\|\pi\|\le 1\}$ is compact in the $\sigma(L^*,L)$-topology.

Let $(A,\mathcal{A},\nu)$ be a finite measure space. A map $f:A\to\ell^\infty$ is said to be is weak*-measurable if for each $\boldsymbol{p}\in\ell^1$, $\boldsymbol{p}f(a)$ is measurable. A weak*-measurable map $f(a)$ is said to be Gel'fand integrable if there exists an element $\xi \in \ell^\infty$ such that for each $\boldsymbol{p} \in \ell^1$, $\boldsymbol{p}\xi = \int \boldsymbol{p} f(a) d\nu$. The vector ξ is denoted by $\int f(a) d\nu$ and called Gel'fand integral of f.

Fact 4 (Diestel and Uhl (1977, pp.53-4)). If $f : A \to \ell^\infty$ is weak*-continuous and $\boldsymbol{p} f(a)$ is integrable function for all $\boldsymbol{p} \in \ell^1$, then f is Gel'fand integrable.

Let S be a complete and separable metric space. We denote the set of all closed subsets of a set S by $\mathcal{F}(S)$. The topology τ_c on $\mathcal{F}(S)$ of closed convergence is a topology which is generated by the base

$$[K; G_1 \dots G_n] = \{F \in \mathcal{F}(S) | F \cap K = \emptyset, F \cap G_i \neq \emptyset, i = 1 \dots n\}$$

as K ranges over the compact subsets of S and G_i are arbitrarily finitely many open subsets of S. It is well known that if X is locally compact separable metric space, then $\mathcal{F}(X)$ is compact and metrizable. Moreover, a sequence F_n converges to $F \in \mathcal{F}(S)$ if and only if $Li(F_n) = F = Ls(F_n)$, where $Li(F_n)$ denotes the topological limes inferior of $\{F_n\}$ which is defined by

$\xi \in Li(F_n)$ if and only if there exists an integer N and a sequence $\xi_n \in F_n$ for all $n \geq N$ and $\xi_n \to \xi \ \ (n \to \infty)$,

and $Ls(F_n)$ is the topological limes superior which is defined by

$\xi \in Ls(F_n)$ if and only if there exists a sub-sequence F_{n_q} with $\xi_{n_q} \in F_{n_q}$ for all q and $\xi_{n_q} \to \xi \ \ (q \to \infty)$,

see Hildenbrand (1974, pp.15-19) for details.

The next Facts are also well known for the mathematical economics.

Fact 5 (Fatou's lemma in ℓ dimensions, Hildenbrand (1974), p.69). Let $(\phi)_{n \in \mathbb{N}}$ be a sequence of integrable functions of a measure space $(\Omega, \mathcal{A}, \mu)$ to $\mathbb{R}^\ell_+$. Suppose that $\lim \int_\Omega \phi_n d\mu$ exists. Then there exists an integrable function $\phi : (\Omega, \mathcal{A}, \mu) \to \mathbb{R}^\ell_+$ such that

(*a*) $\phi(\omega) \in Ls(\phi_n(\omega))$ a.e. in A

(*b*) $\int_\Omega \phi(\omega) d\mu \leq \lim_{n \to \infty} \int_\Omega \phi_n(\omega) d\mu$.

Fact 6 (Hildenbrand (1974, Theorem 6, p.68). Let $(\phi)_{n \in \mathbb{N}}$ be a sequence of measurable correspondences of a measure space $(\Omega, \mathcal{A}, \mu)$ to a bounded subset of $\mathbb{R}^\ell_+$. Then $Ls(\int_A \phi(a) d\mu) \subset \int_A (Ls(\phi(a)) d\mu$.

Fact 7 (Hildenbrand (1974, (2) and (37))). Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and (S, d) be a separable metric space. If f_n, f are (Borel) measurable functions from A to S and $d(f_n(a), f(a)) \to 0$ a.e. in μ, then $\mu \circ f_n^{-1} \to \mu \circ f^{-1}$.

2.2 The Description of the Economy

We now describe our economy in this paper. Let $\beta > 0$ be a given positive number, and ℓ be a positive integer. We will assume that the consumption set X of each consumer is the set of nonnegative vectors whose coordinates after ℓ are bounded by β,

$$X = \{\xi = (\xi^t) \in \ell^\infty |\ 0 \le \xi^t \text{ for } t \ge 1,\ \xi^t \le \beta \text{ for } t > \ell\}.$$

Of course the $\beta > 0$ is intended to be a very large number. We call the first ℓ commodities, $x^1, x^2 \dots x^\ell$ the primary commodities. Then it is obvious that the consumption set is written as

$$X = P \times Z,$$

where $P = \mathbb{R}^\ell_+$ and $Z = \{\boldsymbol{z} = (z^t) \in \ell^\infty |\ 0 \le z^t \le \beta \text{ for all } t \ge 1\}$. From now on, we will sometimes denote $\xi = (\boldsymbol{x}, \boldsymbol{z}) \in P \times Z$ for $\xi \in X$. From Fact 2, we have $\tau_d = \sigma(\ell^\infty, \ell^1) = \tau(\ell^\infty, \ell^1)$ on the set Z. Since Z is compact in τ_d (hence $\sigma(\ell^\infty, \ell^1)$ and $\tau(\ell^\infty, \ell^1)$) topology, $X = \mathbb{R}^\ell \times Z$ is locally compact separable metric space. Hence $\mathcal{F}(X \times X)$ is a compact metric space, so that it is complete and separable.

Remark. This type of a consumption set already appeared in Mas-Colell (1975) in which the consumption set was assumed to be $\mathbb{R}_+ \times M$, where M is a bounded subset of $\mathcal{M}(K)$, the space of measures on a compact metric space K. He also assumed that the measures in M are integer valued, and an element of $\mathbb{R}_+$ the homogeneous good. We will discuss a similar situation in Section 4.

Let $\mathcal{P} \subset \mathcal{F}(X \times X)$ be the collection of allowed preference relations which will be assumed to satisfy the following assumptions.

(i) $\succsim \in \mathcal{P}$ is complete, transitive and reflexive,

(ii) (local non-satiation) for each $\xi \in X$ and every neighborhood U of ξ, there exists $\zeta \in U$ such that $\xi \prec \zeta$, where $\xi \prec \zeta$ means that $(\xi, \zeta) \notin \succsim$,

(iii) (overriding desirability of the primary commodities) for each ξ and $\zeta \in X$ and every $t = 1 \dots \ell$, there exists $\delta > 0$ such that $\xi \prec \zeta + \delta \boldsymbol{e}_t$, where $\boldsymbol{e}_t = (0 \dots 0, 1, 0 \dots)$ and 1 is in the t-th coordinate.

Note that preferences are τ_d (hence $\sigma(\ell^\infty, \ell^1)$ and $\tau(\ell^\infty, \ell^1)$) continuous, since $\mathcal{P} \subset \mathcal{F}(X \times X)$.

An initial endowment is assumed to be nonnegative vectors ω of ℓ^∞, or $\omega \in \ell^\infty_+$. We will restrict the set of Ω of all allowed endowments is of the form

$$\Omega = \{\omega = (\omega^t) \in \ell^\infty |\ 0 \le \omega^t \le \gamma,\ t \in \mathbb{N}\}$$

for some fixed positive number $\gamma < \beta$. It is τ_d-compact subset of ℓ^∞, hence $\sigma(\ell^\infty, \ell^1)$-compact by the same reason of the set Z. For $\xi \in X = P \times Z$, let $\xi = (\boldsymbol{x}, \boldsymbol{z})$, $\boldsymbol{x} \in P$ and $\boldsymbol{z} \in Z$. Correspondingly, we denote $\Omega = \Omega_P \times \Omega_Z$ and $\omega = (\omega_P, \omega_Z) \in \Omega_P \times \Omega_Z$.

Let $(A, \mathcal{A}, \nu)$ be a probability space of the consumers which is assumed to be atomless.

Definition 1. An economy is a Borel measurable mapping $\mathcal{E} : A \to \mathcal{P} \times \Omega, \;\; a \mapsto (\succsim_a, \omega(a))$.

Since Ω is compact, it follows from Fact 4 that there exists the Gel'fand integral $\int_A \omega(a) d\nu$ of the map $\omega : A \to \Omega$. For $\xi = (\xi^t) \in \mathbb{R}^\ell$ or ℓ^∞, $\xi \geq \mathbf{0}$ means that $\xi^t \geq 0$ for all t and $\xi > 0$ means that $\xi \geq \mathbf{0}$ and $\xi \neq \mathbf{0}$. $\xi \gg \mathbf{0}$ means that $\xi^t > 0$ for all t. Finally for $\xi = (\xi^t) \in \ell^\infty$, we denote by $\xi \ggg \mathbf{0}$ if and only if there exists an $\epsilon > 0$ such that $\xi^t \geq \epsilon$ for all t.

The following assumptions are well known and will be used in order to make every consumer's income to be positive.

Assumption (E). $\int_A \omega(a) d\nu \ggg \mathbf{0}$,

Assumption (P). Denoting $\omega = (\omega_P, \omega_Z) \in \Omega$, $\nu(\{a \in A | \;\omega_P(a) > \mathbf{0}\}) = 1$.

Definition 2. A pair $(\boldsymbol{p}, \xi)$ of a price vector $\boldsymbol{p} \in \ell^1_+$ and an integrable map $\xi : A \to X$ is called a competitive equilibrium of the economy $\mathcal{E}$ if the following conditions hold,

(E-1) $\boldsymbol{p}\xi(a) \leq \boldsymbol{p}\omega(a)$ and $\xi(a) \succsim_a \zeta$ whenever $\boldsymbol{p}\zeta \leq \boldsymbol{p}\omega(a)$ a.e,

(E-2) $\int_A \xi(a) d\nu \leq \int_A \omega(a) d\nu$.

The condition (E-1) says that almost all consumers maximize their utilities under their budget constraints. In the condition (E-2) which says that the total demand is equal to the total endowment (the market condition), the Gel'fand integral $\int_A \boldsymbol{z}(a) d\nu$ exists by virtue of Fact 4 and $\int_A \xi(a) d\nu$ is equal to $(\int_A \boldsymbol{x}(a) d\nu, \int_A \boldsymbol{z}(a) d\nu)$, where $\int_A \boldsymbol{x}(a) d\nu$ is the usual Lebesgue integral of $\boldsymbol{x}$ with respect to the measure ν.

The main result of this paper now reads

Theorem 1. Let $\mathcal{E}$ be an economy which satisfies the assumptions (E) and (P). Then there exists a competitive equilibrium $(\boldsymbol{p}, \xi)$ for $\mathcal{E}$.

3 Proof of Theorem 1

Let $\mathcal{E} : A \to \mathcal{P}\times\Omega$ be the economy. For each $n \in \mathbb{N}$, let K^n be the canonical projection of ℓ^∞ to $\mathbb{R}^n$, $K^n = \{\xi = (\xi^t) \in \ell^\infty | \xi = (\xi^1, \xi^2 \dots \xi^n, 0, 0 \dots)\}$. Naturally we can identify K^n with $\mathbb{R}^n$, or $K^n \approx \mathbb{R}^n$. Recall $P = \mathbb{R}^\ell_+$ and define

$$X^n = P \times (Z \cap K^n), \;\; \mathcal{P}^n = \mathcal{P} \cap \mathcal{F}(X^n \times X^n), \;\; \Omega^n = \Omega_P \times (\Omega_Z \cap K^n),$$

and for every $\succsim \in \mathcal{P}$ and $\omega = (\omega^1, \omega^2 \dots \omega^{\ell+n}, \omega^{\ell+n+1} \dots) \in \Omega$, we denote $\succsim^n = \succsim \cap (X^n \times X^n) \in \mathcal{P}^n$ and $\omega_n = (\omega^1, \omega^2 \dots \omega^{\ell+n}, 0, 0 \dots) \in \Omega^n$, the canonical projection of $\succsim$ and ω, respectively. They induce finite dimensional economies $\mathcal{E}^n : A \to \mathcal{P}^n \times \Omega^n$ defined by $\mathcal{E}^n(a) = (\succsim^n_a, \omega_n(a))$, $n = 1, 2 \dots$. We have

Lemma 1. $\mathcal{E}^n(a) \to \mathcal{E}(a)$ a.e.

Proof. See Appendix II. ∎

Lemma 2. For each n, there exists a competitive equilibrium $(\pi_n, \xi_n(a))$ for the economy $\mathcal{E}^n$.

Proof. See Theorem A1 in Appendix I. ∎

Then for each n, there exist a price vector $\pi_n \in \mathbb{R}^{\ell+n}_+$ and an allocation $(\xi_n(a))$ which satisfy

$$\pi_n \xi_n(a) \le \pi_n \omega_n(a) \;\text{ and }\; \xi_n(a) \succsim^n_a \zeta \;\text{ whenever }\; \pi_n \zeta \le \pi_n \omega_n(a) \;\text{ a.e,}$$

and

$$\int_A \xi_n(a) d\nu \le \int_A \omega_n(a) d\nu.$$

In the following, we will often denote $\xi_n(a) = (\boldsymbol{x}_n(a), \boldsymbol{z}_n(a)) \in P \times Z^n$, where $Z^n = Z \cap K^n$.

By the Fatou's lemma in ℓ-dimension (Fact 5), we have an integrable function $\boldsymbol{x} : A \to \mathbb{R}^\ell_+$ such that $\boldsymbol{x}_n(a) \to \boldsymbol{x}(a)$ a.e. and that $\int_A \boldsymbol{x}(a) d\nu \le \int_A \omega_P(a) d\nu$. For each $a \in A$, $\boldsymbol{z}_n(a) \in Z$ and Z is compact in the weak*-topology, hence we have $\boldsymbol{z}(a) \in Z$ with $\boldsymbol{z}_n(a) \to \boldsymbol{z}(a)$ a.e. Consequently we have $\xi_n(a) \to \xi(a) \in X$ a.e. We can show that

Lemma 3. $\boldsymbol{z}(\cdot)$ is Gel'fand integrable.

Proof. See Appendix II. ∎

It follows from Lemma 1, Lemma 3 and $\int_A \xi_n(a) d\nu \le \int_A \omega_n(a) d\nu$ for all n that

$$\int_A \xi(a) d\nu = \lim_{n\to\infty} \int_A \xi_n(a) d\nu \le \lim_{n\to\infty} \int_A \omega_n(a) d\nu = \int_A \omega(a) d\nu,$$

in which the first and the second equality follow from

Lemma 4. Let $\{f_n\}$ be a sequence of Gel'fand integrable functions from A to ℓ^∞ which converges a.e to f in the weak*-topology. Then it follows that $\int_A f_n(a)d\nu \to \int_A f(a)d\nu$ in the weak* topology.

Proof. See Appendix II. ∎

Without loss of generality, we can assume that $\pi_n \mathbf{1} = \sum_{t=1}^{\ell+n} p_n^t = 1$ for all n, where $\pi_n = (p_n^t)$ and $\mathbf{1} = (1, 1 \dots)$. Here we have identified $\pi_n \in \mathbb{R}^{\ell+n}$ with a vector in ℓ^1 which is also denoted by π_n as $\pi_n = (\pi_n, 0, 0 \dots)$. Since the set $\Delta = \{\pi \in ba_+ |\ \|\pi\| = \pi \mathbf{1} \leq 1\}$ is weak* compact by the Alaoglu's theorem (Fact 3), we have a price vector $\pi = (\pi_P, \pi_Z)$, $\pi_P = (p_P^t) \in \mathbb{R}_+^\ell$, $\pi_Z \in ba_+$ with $\sum_{t=1}^{\ell} p_P^t + \pi_Z \mathbf{1} = 1$ and such that $\pi_n \to \pi$ in the $\tau(\mathbb{R}^\ell) \times \sigma(ba, \ell^\infty)$-topology, where $\tau(\mathbb{R}^\ell)$ is the usual topology for $\mathbb{R}^\ell$.

Lemma 5. $\pi_P \gg \mathbf{0}$.

Proof. Suppose not. Then we have $p_n^t \to p_P^t = 0$ for some $t = 1 \dots \ell$. For $\omega = (\omega^t) \in \Omega$, let $\omega^{-t} = (\omega^1 \dots \omega^{t-1}, 0, \omega^{t+1} \dots)$. Since $\pi \int_A \omega^{-t}(a)\nu = \pi \int_A \omega(a)\nu > 0$ by Assumption (E), it follows that $\pi\omega^{-t}(a) > 0$ on a set with positive ν-measure, hence $\pi_n \omega_n^{-t}(a) > 0$ for n large enough on a set with positive ν-measure, since $\pi_n \omega_n^{-t}(a) = \pi_n \omega^{-t}(a) \to \pi\omega^{-t}(a)$. (Recall that $\pi_n = (p^1 \dots p^{\ell+n}, 0 \dots)$ and $\omega_n^{-t}(a)$ is the projection of $\omega^{-t}(a) \in \ell^\infty$ to $\mathbb{R}^{\ell+n}$.) Setting $p_n^t \delta_n = \pi_n \omega_n^{-t}(a)$, we now define

$$\zeta_n = \delta_n \boldsymbol{e}_t + \omega_n(a) - \omega_n^{-t}(a),$$

where $\boldsymbol{e}_t = (0 \dots 0, 1, 0 \dots)$ and 1 is at the t-th coordinate. Then we have $\pi_n \zeta_n = \pi_n \omega_n(a)$ and $\delta_n \to \infty$. We now claim that

$$\xi_n(a) \prec_a^n \zeta_n \text{ for } n \text{ sufficiently large,}$$

which yields a contradiction. Indeed, the set $\{\xi_n(a)|\ n \in \mathbb{N}\}$ is contained in a compact subset C of X, and for each $\xi \in C$, one obtains $\delta_\xi > 0$ such that

$$\xi \prec_a \delta_\xi \boldsymbol{e}_t + \omega(a) - \omega^{-t}(a).$$

Since $\prec_a$ is continuous, there exists a neighborhood U_ξ of ξ and an $N_1 \in \mathbb{N}$ such that

$$\xi' \prec_a \delta_\xi \boldsymbol{e}_t + \omega_n(a) - \omega_n^{-t}(a)$$

for every $\xi' \in U_\xi$ and all $n \geq N_1$. Since C is compact, we can take $\xi_1 \dots \xi_{N_2}$ such that $C \subset \cup_{i=1}^{N_2} U_{\xi_i}$. Take an $n \geq N_1$ such that $\delta_n > max\{\delta_{\xi_1} \dots \delta_{\xi_{N_2}}\}$. Then $\xi_n(a) \prec_a^n \zeta_n$ as desired. ∎

We now prove that

Lemma 6. $\xi(a) \prec_a \zeta$ implies that $\pi\omega(a) < \pi\zeta$ a.e.

Proof. Suppose not. Then there exists $\zeta = (\zeta^t) \in X$ such that $\pi\zeta \leq \pi\omega$ and $\xi(a) \prec_a \zeta$ on a set with ν-positive measure. By Lemma 4 and Assumption (P), we can assume that $\pi\omega(a) > 0$ a.e. Since the preferences are continuous, we can assume without loss of generality that $\pi\zeta < \pi\omega(a)$ and $\xi(a) \prec_a \zeta$. Let $\zeta_n = (\zeta^1 \ldots \zeta^{\ell+n}, 0, 0 \ldots)$ be the projection of ζ to X^n. Since $\zeta_n \to \zeta$ in the $\sigma(\ell^\infty, \ell^1)$-topology, we have for sufficiently large N that $\pi\zeta_N \leq \pi\zeta < \pi\omega(a)$ and $\xi(a) \prec_a \zeta_N$, since $\pi \geq \mathbf{0}$ and $\zeta_N \leq \zeta$. Since $\pi_n \to \pi$ and $\xi_n(a) \to \xi(a)$, it follows that for some $n \geq N$, $0 \leq \pi_n\zeta_N < \pi_n\omega(a) = \pi_n\omega_n(a)$ and $\xi_n(a) \prec_a \zeta_N$, or $\xi_n(a) \prec_a^n \zeta_N$. This contradicts the fact that $(\pi_n, \xi_n(a))$ is an equilibrium for $\mathcal{E}^n$. ∎

Let $\pi = \pi_c + \pi_p$ be the Yosida-Hewitt decomposition and denote $\pi_c = \boldsymbol{p}$. Suppose that $\xi(a) \prec_a \zeta$. Then we can assume that $\xi(a) \prec_a \zeta_n$ for n sufficiently large, hence it follows from Lemma 6 that $\pi\zeta_n > \pi\omega(a)$ for n sufficiently large. We will show that $(\boldsymbol{p}, \xi(a))$ is an equilibrium of the economy $\mathcal{E}$. Since π_p is purely finitely additive, $\pi_p(\{1 \ldots n\}) = 0$ for each n. It follows from this and $\pi_c \geq \mathbf{0}$ that

$$\pi\zeta_n = (\pi_c + \pi_p)\zeta_n = \pi_c\zeta_n \leq \pi_c\zeta = \boldsymbol{p}\zeta,$$

since $\zeta_n \leq \zeta$. On the other hand, $\pi_p \geq \mathbf{0}$ and $\omega(a) \geq \mathbf{0}$ imply that $\pi\omega(a) = (\pi_c + \pi_p)\omega(a) \geq \pi_c\omega(a) = \boldsymbol{p}\omega(a)$, and consequently we have $\boldsymbol{p}\zeta > \boldsymbol{p}\omega(a)$. Summing up, we have verified that

$$\xi(a) \prec \zeta \text{ implies that } \boldsymbol{p}\omega(a) < \boldsymbol{p}\zeta \text{ a.e.}.$$

Since the preferences are locally non-satiated, there exists $\zeta \in X$ arbitrarily close to $\xi(a)$ such that $\xi(a) \prec \zeta$, therefore we have

$$\boldsymbol{p}\xi(a) \geq \boldsymbol{p}\omega(a) \text{ a.e.}$$

On the other hand, it follows from $\int_A \xi(a)d\nu \leq \int_A \omega(a)d\nu$ that

$$\int_A \boldsymbol{p}\xi(a)d\nu = \boldsymbol{p}\int_A \xi(a)d\nu \leq \boldsymbol{p}\int_A \omega(a)d\nu = \int_A \boldsymbol{p}\omega(a)d\nu.$$

Therefore $\boldsymbol{p}\xi(a) = \boldsymbol{p}\omega(a)$ a.e. ∎

4 The Case of Indivisible Commodities

In this section, we discuss a model of large exchange economies containing the commodities which are consumed in the integer unit, or the indivisible commodities.

We still assume that the primary commodities are divisible, hence $\xi^t \in \mathbb{R}$ for $t = 1 \dots \ell$, but the commodities $t \geq \ell + 1$ are indivisible, so that $\xi^t \in \mathbb{N}$ for $t \geq \ell + 1$. The consumption set X_{idv} of each consumer in this section is then given by

$$X_{idv} = \{\xi = (\xi^t) \in \ell^\infty | \ 0 \leq \xi^t < +\infty \ \text{ for } \ 1 \leq t \leq \ell, \xi^t \in \mathbb{N}, \xi^t \leq \beta \ \text{ for } \ t \geq \ell + 1\}.$$

As in the previous section, X_{idv} is isomorphic to $\mathbb{R}^\ell_+ \times Z_{idv} = P \times Z_{idv}$, where $Z_{idv} = \{\xi = (\xi^t) \in \ell^\infty_+ | \ \xi^t \in \mathbb{N}, \xi^t \leq \beta \ \text{ for all } \ t\}$. Similarly we define the set of allowed endowments Ω_{idv} by

$$\Omega_{idv} = \{\omega = (\omega^t) | \ 0 \leq \omega^t \leq \gamma \ \text{ for all } \ t, \omega^t \in \mathbb{N} \ \text{ for } \ t \geq \ell + 1\}$$

and we sometimes write $\Omega_{idv} = \Omega_P \times \Omega_{idv,Z} \subset \mathbb{R}^\ell_+ \times Z_{idv}$.

The space of allowed preferences $\mathcal{P}_{idv} \subset \mathcal{F}(X_{idv} \times X_{idv})$ is defined as a compact set of continuous preorders which satisfy the conditions (i),(ii) and (iii) of the previous section. An economy $\mathcal{E}_{idv}$ is a (Borel) measurable map from $(A, \mathcal{A}, \nu)$ to $(\mathcal{P}_{idv} \times \Omega_{idv}, \mathcal{B}(\mathcal{P}_{idv} \times \Omega_{idv}))$. The competitive equilibrium $(\boldsymbol{p}, \xi)$ for the economy $\mathcal{E}_{idv}$ is defined as in the same way as that of the previous section. In order to handle the indivisible commodities on the infinite dimensional commodity space, we need the extra condition. First we require the primary commodities are suitably "diffused" or "dispersed".

Assumption (D). $\nu(a \in A | \ \sum_{t=1}^{\ell} p^t \omega^t_P(a) = w\}) = 0$ for every $\boldsymbol{p} = (p^t) \in \mathbb{R}^\ell_+$ with $\boldsymbol{p} \neq \boldsymbol{0}$ and every $w \in \mathbb{R}$.

On the other hand, we need that the variations of the indivisible endowments of the economies are very small.

Assumption (F). $\omega_Z : A \to \Omega_Z$ is a simple function.

We now state an equilibrium existence theorem with indivisible commodities.

Theorem 2. Let $\mathcal{E}_{idv}$ be an economy. Suppose that $\mathcal{E}_{idv}$ satisfies the assumptions (E), (P), (D) and (F). Then there exists a competitive equilibrium $(\boldsymbol{p}, \xi)$ for $\mathcal{E}_{idv}$.

Remark. The assumption (D) implies that the distribution of the primary commodities are "dispersed", namely that the distribution of their market values does not give a positive measure on any specific value for each price system. It was first introduced by Mas-Colell (1977) and generalized in this form by Yamazaki (1978). The role of this condition is well known. By the indivisible commodities, the behavior of the individual demand generally exhibits the discontinuity at some some price vector. However, by virtue of the assumption (D), the mass of the discontinuous consumers will be 0 at each price vector, hence the aggregate demand preserves the upper hemi-continuity. The assumption (F) is really a very strong assumption. Fortunately, however, Theorem

3 shows that the set of economies satisfying the assumption (F) is a dense subset of the set of all economies.

Proof of Theorem 2. The proof is almost the same as that of Theorem 1, so that we shall only give a brief outline. From now on, we suppress idv for simplicity.

Let $\mathcal{E}^n(a) = (\succsim_a^n, \omega_n(a))$ be the finite dimensional sub-economies converging almost everywhere to $\mathcal{E}(a) = (\succsim_a, \omega(a))$. The economy $\mathcal{E}^n : A \to \mathcal{P}^n \times \Omega^n$ satisfies the assumptions of Theorem A2 in Appendix I, hence there exists a competitive equilibrium $(\pi_n, \xi_n(a))$ for each n. Then as in the proof of Theorem 1, we can show that the sequences (π_n), $(\xi_n(a))$ converges to $\pi \in ba$, $(\xi(a))$ respectively, or $\pi_n \to \pi$ in the $\tau(\mathbb{R}^\ell) \times \sigma(ba, \ell^\infty)$-topology and $\xi_n(a)$ to $\xi(a) \in X$ a.e in the $\sigma(\ell^\infty, \ell^1)$-topology, respectively. By exactly the same argument of the proof of Theorem 1, we can also show that

$$\int_A \xi(a) d\nu \leq \int_A \omega(a) d\nu.$$

Denote $\pi = (\pi_P, \pi_Z) \in \mathbb{R}_+^\ell \times ba$. We can show Lemma 4 also holds here, hence $\pi_P \gg \mathbf{0}$.

We have only to take care of the indivisibilities to show Lemma 6, or namely that

$$\xi(a) \prec_a \zeta \text{ implies that } \pi\omega(a) < \pi\zeta \text{ a.e.}$$

Suppose not. Then there exists $\zeta = (\zeta^t) \in X$ such that $\pi\zeta \leq \pi\omega(a)$ and $\xi(a) \prec_a \zeta$ on a set of ν-positive measure. Without loss of generality, we can assume that $\pi\zeta_n \leq \pi\omega(a)$ and $\xi(a) \prec_a \zeta_n$ for some n, where ζ_n is the projection of ζ to $\mathbb{R}^{\ell+n}$, or $\zeta_n = (\zeta^1 \dots \zeta^{\ell+n}, 0, 0 \dots)$. We rewrite ζ_n as $\zeta_n = (\zeta_P, \zeta_{Z,n})$, $\zeta_P = (\zeta^1 \dots \zeta^\ell)$ and $\zeta_{Z,n} = (\zeta^{\ell+1} \dots \zeta^{\ell+n}, 0, 0 \dots)$. $\pi_P \gg \mathbf{0}$ and the assumption (P) imply that $\pi\omega(a) > 0$ a.e. When $\pi\zeta_n < \pi\omega$, we can show exactly in the same way as in the proof of theorem 1 that $\pi_N\zeta_n < \pi_N\omega_N(a)$ and $\xi_N(a) \prec_a^N \zeta_n$ for some $N \geq n$, a contradiction.

If $\pi\zeta_n = \pi\omega(a)$, then we can assume that $\zeta^t > 0$ for some $t = 1 \dots \ell$. Indeed, if $\zeta^t = 0$ for all $t = 1 \dots \ell$, it follows from the assumption (D) and $\pi_P = (p_P^t) \gg \mathbf{0}$ that

$$\nu\left(\left\{a \in A \middle|\ \pi(\mathbf{0}, \zeta_{Z,n}) = \pi\omega(a)\right\}\right) = \nu\left(\left\{a \in A \middle|\ \sum_{t=1}^{\ell} p_P^t \omega_P^t(a) = \pi_Z(\zeta_{Z,n} - \omega_Z)\right\}\right) = 0,$$

since the set $\{\pi_Z(\zeta_{Z,n} - \omega_Z)\}$ is countable by the assumption (F) and the fact that the set of the sequences of the form

$$\{\zeta_{Z,n} = (\zeta^{\ell+1} \dots \zeta^{\ell+n}, 0 \dots) |\ \zeta^t (\leq \beta) \in \mathbb{N},\ t = \ell+1 \dots \ell+n,\ n = 1, 2. \dots\}$$

is countable. Then we have a sequence $(\hat{\zeta}_i)$ with $\hat{\zeta}_i < \hat{\zeta}_{i+1} < \cdots < \to \zeta_n$, $\hat{\zeta}_i \in \mathbb{R}_+^{\ell+n}$ for all i. Since the preferences are continuous, $\pi\hat{\zeta}_i < \pi\omega(a)$ and $\xi(a) \prec_a \hat{\zeta}_i$ for i large enough. Hence this case is reduced to the first case, and leads to a contradiction.

Let $\pi = \pi_p + \pi_c = \pi_p + \boldsymbol{p}$ be the Yosida-Hewitt decomposition. Then as in the proof of Theorem 1 we can show that

$$\boldsymbol{p}\xi(a) \leq \boldsymbol{p}\omega \text{ and } \xi(a) \succsim_a \zeta \text{ whenever} \boldsymbol{p}\zeta \leq \boldsymbol{p}\omega(a) \text{ a.e.}$$

This completes the proof. ∎

Although the assumption (F) is very strong, we can show that there are "many" economies which satisfy the assumption (F) in the following sense.

Let $\mathscr{E}$ be the set of all economies, or $\mathscr{E} = \{\mathcal{E} : A \to \mathcal{P} \times \Omega)|\ \mathcal{E}$ is a Borel Measurable map.$\}$. Hildenbrand (1974) introduced a topology on the space of economies $\mathscr{E}$ in the following way.

According to him, a sequence of economies $\{\mathcal{E}_n = (\succsim_a^n, \omega^n(a))\}$ converges to an economy $\mathcal{E} = (\succsim_a, \omega(a))$ if and only if $\nu \circ \mathcal{E}_n^{-1} \to \nu \circ \mathcal{E}^{-1}$ in the weak topology of the measures, and $\int_A \omega^n(a) d\nu \to \int_A \omega(a) d\nu$. In other words, he introduces the metric $d_{\mathscr{E}}$ on the space $\mathscr{E}$ defined by $d_{\mathscr{E}}(\mathcal{E}, \mathcal{E}') = \rho(\nu \circ \mathcal{E}, \nu \circ \mathcal{E}') + |\int_A \omega d\nu - \int_A \omega' d\nu|$, where ρ is the Prohorov metric on $\mathcal{M}(\mathcal{P} \times \Omega)$ (Hildenbrand (1974, p.49).

Then we can prove that

Theorem 3. Let $\mathscr{E}_F$ be the set of economies which satisfy the assumption (F). $\mathscr{E}_F$ is a dense subset of $\mathscr{E}$.

Proof. First we prove that

Lemma 7. There exists a sequence of simple functions $\{\omega_Z^n(a)\}$ with $\omega_Z^n(a) \in \Omega_{idv,Z}$ and $\omega_Z^n(a) \to \omega_Z(a)$ a.e in the weak*-topology.

Proof. appendix II. ∎

It follows from Lemma 7 that we can take a sequence $\{\omega^n(a) = (\omega_P(a), \omega_Z^n(a))\}$ with $\omega^n(a) \to \omega(a)$ a.e in the weak*-topology. Define $\mathcal{E}_n(a) = (\succsim_a, \omega^n(a))$. Then by fact 6, we have

$$\nu \circ \mathcal{E}_n^{-1} \to \nu \circ \mathcal{E}^{-1},$$

and it follows from Lemma 4 that $\int_A \omega^n(a) d\nu \to \int_A \omega(a) d\nu$. ∎

Appendix I

Since we did not find elsewhere finite dimensional equilibrium existence theorems with the forms which we need in the text, we will give the proofs of them for the completeness. The consumption set of each consumer is $X = \mathbb{R}_+^\ell \times Z$, where

$$Z = \{\boldsymbol{z} = (z^t) \in \mathbb{R}_+^n \,\big|\; z^t \le \beta, t = 1 \dots n\}.$$

A preference relation $\succsim \subset X \times X$ is a complete and transitive binary relation which is closed relative to $X \times X$, satisfies the local non-satiation and the overriding desirability of the primary commodities. Recall that we denote by $\mathcal{P}$ the set of all allowed preference relations endowed with the topology of closed convergence. Let $(A, \mathcal{A}, \nu)$ be the atomless measure space, and let $\omega : A \to \Omega = \mathbb{R}_+^\ell \times \mathbb{R}_+^n$ be an integrable map,

$$\omega : a \mapsto \omega(a), \quad \int_A \omega(a) d\nu < +\infty,$$

which assigns the consumer a his/her endowment vector. An economy $\mathcal{E}$ is a measurable map of A to $\mathcal{P} \times \Omega$,

$$\mathcal{E} : a \mapsto (\succsim_a, \omega(a)) \in \mathcal{P} \times \Omega.$$

A feasible allocation is an integrable map f of A to X such that $\int_A f(a) d\nu \le \int_A \omega(a) d\nu$. A pair of a price vector $\pi \in \mathbb{R}_+^{\ell+n}$ and a feasible allocation (π, f) is competitive equilibrium of $\mathcal{E}$ if $\pi f(a) \le \pi\omega(a)$ and $f(a) \succsim_a \xi$ whenever $\pi\xi \le \pi\omega(a)$ a.e. in A.

Theorem A1. An economy $\mathcal{E}$ has an equilibrium if (i) $\int_A \omega(a) \gg \mathbf{0}$ and satisfies

$$\text{(ii)} \;\; \nu\Big(\{a \in A |\;\; \omega^t(a) > 0 \;\text{ for some }\; t = 1 \dots \ell\}\Big) = 1.$$

Proof. Let $b = \int_A \sum_{t=1}^{\ell+n} \omega^t(a) d\nu$. Then $b > 0$. For $k = 1, 2, \dots$, define $A_k = \{a \in A |\;\; \omega(a) \le kb\mathbf{1}\}$, where $\mathbf{1} = (1 \dots 1) \in \mathbb{R}^{\ell+n}$. Obviously $\emptyset \ne A_1 \subset A_2 \subset \dots$. Let $\mathcal{A}_k = \{C \cap A_k |\;\; C \in \mathcal{A}\}$ and ν_k be the restriction of ν to $\mathcal{A}_k$. Since $(A, \mathcal{A}, \nu)$ is atomless, so is $(A_k, \mathcal{A}_k, \nu_k)$ for each k. We then define for each k the (truncated) consumption set by $X_k = \{\xi \in X |\;\; \xi \le kb\mathbf{1}\}$ and the (truncated) demand correspondence

$$\phi_k(a, \pi) = \left\{\xi \in X_k \,\middle|\; \pi\xi \le \pi\omega(a), \;\text{ and if }\; \pi\zeta \le \pi\omega(a), \;\text{ then }\; \xi \succsim_a \zeta\right\},$$

where $\pi \in \Delta = \{\pi = (\pi^t) |\;\; \sum_{t=1}^{\ell+n} \pi^t = 1\}$. The quasi-demand correspondence is defined by

$$\psi_k(a, \pi) = \begin{cases} \phi_k(a, \pi) & \text{if } \pi\omega(a) > 0, \\ \{\xi \in X_k |\;\; \pi\xi = 0\} & \text{otherwise.} \end{cases}$$

Since X_k is compact, it is standard to verify that $\phi_k(a,\pi) \neq \emptyset$, and it follows from Corollary 2 of Hildenbrand (1974, p.104), ψ_k is closed relation, hence it has a measurable graph. We then define the (truncated) mean demand

$$\Phi_k(\pi) = \int_{A_k} \psi_k(a,\pi) d\nu_k \ \text{ for } \ \pi \in \Delta.$$

It is well known that the mean excess demand correspondence $\zeta_k(\pi) = \Phi_k(\pi) - \int_{A_k} \omega(a) d\nu$ is compact and convex valued (Hildenbrand (1974, p.62)). It is upper hemi-continuous (Hildenbrand (1974, Proposition 8, p.73), satisfies the Walras law: $\pi\zeta_k(\pi) \leq 0$ for every $\pi \in \Delta$.

We can then apply the fixed point theorem

Lemma A1 (Hildenbrand (1974, p.39)). Let C be a closed convex cone with the vertex $\mathbf{0}$ in $\mathbb{R}^{\ell+n}$ which is not a linear subspace. If the correspondence ζ of C into $\mathbb{R}^{\ell+n}$ is nonempty, compact and convex valued and upper hemi-continuous, and satisfies $\pi\zeta \leq 0$ for every $\pi \in C$, then there exists $\pi^* \in C$ with $\pi^* \neq \mathbf{0}$ such that $\zeta(\pi^*) \in polar(C)$, where $polar(C)$ is the polar of the set C,

$$polar(C) = \{\zeta \in \mathbb{R}^{\ell+n} | \ \pi\zeta \leq 0 \ \text{ for all } \ \pi \in C\}.$$

Then for each k, there exists a price vector $\pi_k \in \Delta$ and an integrable function $f_k(\cdot)$ of A_k to $\mathbb{R}^{\ell+n}$ such that

$$f_k(a) \in \psi_k(a,\pi_k) \ \text{ a.e. in } \ A_k \tag{1}$$

$$\int_{A_k} f_k(a) d\nu_k \leq \int_{A_k} \omega(a) d\nu_k, \ \ k = 1, 2 \ldots. \tag{2}$$

We extend the domain A_k of f_k to A by defining $f_k(a) = \omega(a)$ for $a \in A \backslash A_k$. Then the condition (2) is replaced by

$$\int_A f_k(a) d\nu \leq \int_A \omega(a) d\nu, \ \ k = 1, 2 \ldots. \tag{3}$$

Since $\pi_k \in \Delta$, we can assume that $\pi_k \to \pi \in \Delta$. Since $f_k(a) \in X$ a.e. in A and the set X is bounded from below, it follows from (3) that the sequence $(\int_A f_k(a) d\nu)$ is bounded. Then by the Fatou's lemma in ℓ-dimensions (Fact 5) that there exists an integrable function f of A to $\mathbb{R}^\ell$ such that

$$f(a) \in Ls(f_k(a)) \ \text{ a.e. in } \ A,$$

$$\int_A f(a) d\nu \leq \int_A \omega(a) d\nu.$$

We complete the proof by showing that

$$f(a) \in \phi(a,\pi) \ \text{ a.e in } \ A.$$

It follows from (1) and $f_k(a) \to f(a)$ that $\pi f(a) \leq \pi\omega(a)$. Now, there exists a positive integer $k(a)$ for a such that

$$k > k(a) \text{ implies } f_k(a) \in \psi_k(a, \pi_k) \text{ a.e. in } A,$$

for we can take $k(a)$ as a positive integer not smaller than $\|\omega(a)\|/b$. Then $0 \leq \omega^t(a)\|\omega(a)\| \leq k(a)b$, $t = 1 \dots \ell + n$.

We claim that $\pi^t > 0$ for $t = 1 \dots \ell$. Suppose not. Then for some $t = 1 \dots \ell$, $\pi^t = 0$. Since $\pi \int_A \omega(a) d\nu > 0$ by the assumption (i), hence $\pi \int_{A_k} \omega(a) d\nu > 0$ for k sufficiently large. Therefore $\pi\omega(a) > 0$ on a set $B \subset A_k (\subset A_{k+1} \subset \cdots \subset A)$ of positive measure for k large enough. Then for $a \in B$, $\psi_k(a, \pi_k) = \phi_k(a, \pi_k)$ for k large enough and $\pi^s\omega^s(a) > 0$ for some $s \neq t$. Take $\delta'_k(a) > 0$ so as to $\pi_k^t \delta'_k(a) - \pi_k^s \omega^s(a) = 0$ and define $\delta_k(a) = min\{\delta'_k(a), bk - \omega^t(a)\}$ and $\zeta_k(a) = \omega(a) + \delta_k(a)\boldsymbol{e}_t - \omega^s(a)\boldsymbol{e}_s$, where $\boldsymbol{e}_t = (0 \dots 0, 1, 0 \dots 0)$ and 1 is in the t-th coordinate. Then we have $\pi_k \zeta_k(a) \leq \pi_k \omega(a)$ for all k large enough and $\delta_k(a) = min\{(\pi_k^s/\pi_k^t)\omega^s(a), bk - \omega^t(a)\} \to +\infty$ as $k \to \infty$, hence $f_k(a) \prec_a \zeta_k(a)$ for k large enough by the overriding desirability of the primary commodities. This contradicts $f_k(a) \in \phi(a, \pi_k)$.

It follows from the assumption (ii) that $\pi\omega(a) > 0$ for almost all $a \in A$. For $\xi \in X$ with $\pi\xi \leq \pi\omega(a)$, we can assume that $\pi\xi < \pi\omega(a)$. Hence $\pi_k\xi < \pi_k\omega(a)$ and $f_k(a) \succsim_a \xi$ for k large enough. By the continuity of $\succsim_a$, we have $f(a) \succsim_a \xi$. ∎

We now consider the model with indivisible commodities. The consumption set of each consumer is $X_{idv} = \mathbb{R}_+^\ell \times Z$, where

$$Z = \{\boldsymbol{z} = (z^t) \in \mathbb{N}^n \Big| \; z^t \leq \beta, t = 1 \dots n\}.$$

A preference relation $\succsim \subset X_{idv} \times X_{idv}$ is a complete and transitive binary relation which is closed relative to $X_{idv} \times X_{idv}$, satisfies the local non-satiation and the overriding desirability of the primary commodities. $\mathcal{P}_{idv}$ is the set of all allowed preference relations endowed with the topology of closed convergence. An economy $\mathcal{E}$ is a measurable map of A to $\mathcal{P}_{idv} \times \Omega_{idv}$,

$$\mathcal{E} : a \mapsto (\succsim_a, \omega(a)) \in \mathcal{P}_{idv} \times \Omega_{idv}.$$

A pair of a price vector $\pi \in \mathbb{R}_+^{\ell+n}$ and a feasible allocation (π, f) is competitive equilibrium of $\mathcal{E}$ if $\pi f(a) \leq \pi\omega(a)$ and $f(a) \succsim_a \xi$ whenever $\pi\xi \leq \pi\omega(a)$ a.e. in A.

Theorem A2. An economy $\mathcal{E}$ has an equilibrium if $\int_A \omega(a) \gg \mathbf{0}$ and satisfies

(i) $\nu\Big(\{a \in A | \; \omega^t(a) > 0 \text{ for some } t = 1 \dots \ell\}\Big) = 1$,

(ii) $\nu\Big(\{a \in A | \; \sum_{t=1}^\ell p^t\omega^t(a) = w\}\Big) = 0$ for every $\boldsymbol{p} = (p^t) \neq \mathbf{0}$ and every $w \in \mathbb{R}$.

Proof. Let $b = \int_A \sum_{t=1}^{\ell+n} \omega^t(a) d\nu > 0$. For $k = 1, 2, \ldots$, define the measure space $(A_k, \mathcal{A}_k, \nu_k)$, the consumption set X_k and the (truncated) demand correspondence $\phi_k(a, \pi)$ as in the proof of Theorem A1. Since X_k is compact, it is standard to verify that $\phi_k(a, \pi) \neq \emptyset$, and Proposition 2 of Hildenbrand (1974, p.102) applies also to the case of indivisible commodities, hence ϕ_k has a measurable graph. We then define the (truncated) mean demand

$$\Phi_k(\pi) = \int_{A_k} \phi_k(a, \pi) d\nu_k \ \text{ for } \ \pi \in \Delta.$$

$\Phi(\cdot)$ is compact and convex valued (Hildenbrand (1974, p.62)). In order to show that it is upper hemi-continuous, we restrict its domain. Define

$$Q_k = \mathbb{R}_-^{\ell+n} \cup \{(-\mathbf{1}_\ell, (1/k)\mathbf{1}_n)\}, \ \ k = 1, 2 \ldots,$$

where $\mathbf{1}_\ell = (1 \ldots 1) \in \mathbb{R}^\ell$ and $\mathbf{1}_n = (1 \ldots 1) \in \mathbb{R}^n$, and let $C(Q_k)$ be a closed convex cone with the vertex $\mathbf{0}$ which is generated by the set Q_k. Let Δ_k be the intersection of its polar set and Δ, or

$$\Delta_k = \{\pi = (\pi^t) \in \mathbb{R}^{\ell+n} | \ \pi\zeta \leq 0 \ \text{ for each } \ \zeta \in C(Q_k)\} \cap \Delta.$$

Obviously $C(Q_1) \supset C(Q_2) \supset \cdots \supset \mathbb{R}_-^{\ell+n}$ and $C(Q_k) \to \mathbb{R}_-^{\ell+n}$ in the topology of closed convergence, namely that $\mathbb{R}_-^{\ell+n} = Li(C(Q_k)) = Ls(C(Q_k))$. Consequently, $\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta$ and $\Delta = Li(\Delta_k) = Ls(\Delta_k)$. Moreover, if $\pi = (\boldsymbol{p}, \boldsymbol{q}) \in \Delta_k$, where $\boldsymbol{p} \in \mathbb{R}^\ell$ and $\boldsymbol{q} \in \mathbb{R}^n$ with $\pi \neq \mathbf{0}$, then $\boldsymbol{p} \gg \mathbf{0}$ for all k. Let k be given. We will show that $\Phi_k : \Delta_k \to X_k$ is upper hemi-continuous. Let (π_n, ξ_n) be a sequence in $\Delta_k \times X_k$ such that $(\pi_n, \xi_n) \to (\pi, \xi) \in \Delta_k \times X_k$ and $\xi_n \in \Phi_k(\pi_n)$ for all n. Since X_k is compact, it suffices to show that $\xi \in \Phi_k(\pi)$. For each n, there exists an integrable function f_n of A_k to X_k with $f_n(a) \in \phi_k(a, \pi_n)$ a.e. in A_k and $\xi_n = \int_{A_k} f_n(a) d\nu$. For the price vector $\pi \in \Delta_k$, define a subset $A_k(\pi)$ of A_k by

$$A_k(\pi) = \{a \in A_k | \ \pi(\mathbf{0}, \zeta) = \pi\omega(a) \ \text{ for some } \ \zeta = (z^t) \in \mathbb{N}^n\}.$$

Since $A_k(\pi) = \{a \in A_k | \ \sum_{t=1}^{\ell} \pi^t \omega^t(a) = \sum_{t=1}^{n} \pi^{\ell+t}(z^t - \omega^{\ell+t}(a))\}$ and $z^t, \omega^{\ell+t}(a) \in \mathbb{N}$ for $t = 1 \ldots n$, it follows from the assumption (ii) that $\nu_k(A_k(\pi)) = 0$, since $\pi^t > 0$, $t = 1 \ldots \ell$. We then claim that

$$Ls(f_n(a)) \subset \phi_k(a, \pi) \ \text{ a.e. in } \ A_k \backslash A_k(\pi). \tag{4}$$

Let $a \in A_k \backslash A_k(\pi)$ be such that $f_n(a) \in \phi_k(a, \pi_n)$ for all n, and let $f_n(a) \to f(a) \in X_k$. Since $\pi_n f_n(a) \leq \pi_n \omega(a)$ for all n, $\pi f(a) \leq \pi\omega(a)$. For $\xi = (x^1 \ldots x^{\ell+n}) \in X_k$ such that $\pi\xi < \pi\omega(a)$, we have $\pi_n \xi < \pi_n \omega(a)$ for all n sufficiently large. By the continuity of $\succsim_a$, one obtains $f(a) \succsim_a \xi$. If $\pi f(a) = \pi\omega(a)$, for some $t = 1 \ldots \ell$, there exist x_i^t $(i = 1, 2, \ldots)$ such that $x_1^t < x_2^t < \cdots \to x^t$, since $a \notin A_k(\pi)$. Let ξ_i be the vector which is equal to $\xi = (x^t)$ but whose t-th coordinate is replaced by x_i^t. Therefore it follows from $\xi_i \to \xi$ that $f(a) \succsim_a \xi$ by the continuity of $\succsim_a$.

The claim (4) combined with Fact (6) then implies that

$$\xi \in Ls\left(\int_{A_k} f_n(a) d\nu_k\right) \subset \int_{A_k} Ls(f_n(a)) d\nu_k \subset \int_{A_k} \phi_k(a, \pi) d\nu_k = \Phi_k(\pi).$$

This proves the upper hemi-continuity of Φ_k.

We can then apply Lemma A1 (the fixed point theorem) to $\zeta_k(\pi) = \int_{A_k} \phi_k(a,\pi) d\nu_k - \int_{A_k} \omega(a) d\nu_k$. Then for each k, there exists a price vector $\pi_k \in \Delta_k$ and an integrable function $f_k(\cdot)$ of A_k to $\mathbb{R}^{\ell+n}$ such that

$$f_k(a) \in \phi_k(a,\pi_k) \ \text{ a.e. in } \ A_k$$
$$\int_{A_k} f_k(a) d\nu_k - \int_{A_k} \omega(a) d\nu_k \in C(Q_k), \ \ k = 1, 2 \ldots. \tag{5}$$

We extend the domain A_k of f_k to A by defining $f_k(a) = \omega(a)$ for $a \in A \backslash A_k$. Then the condition (5) is replaced by

$$\int_A f_k(a) d\nu - \int_A \omega(a) d\nu \in C(Q_k), \ \ k = 1, 2 \ldots. \tag{6}$$

Since $\pi_k \in \Delta_k \subset \Delta$, we can assume that $\pi_k \to \pi \in \Delta$. Since $f_k(a) \in X$ a.e. in A and $X \cap (\int_A \omega(a) d\nu + C(Q_1))$ is a bounded set, it follows from (6) that the sequence $(\int_A f_k(a) d\nu)$ is bounded. Then by the Fatou's lemma in ℓ-dimensions (Fact 5), there exists an integrable function f of A to $\mathbb{R}^{\ell+n}$ such that

$$f(a) \in Ls(f_k(a)) \ \text{ a.e. in } \ A, \tag{7}$$
$$\int_A f(a) d\nu \le \int_A \omega(a) d\nu,$$

since $Ls(C(Q_k)) = \mathbb{R}^{\ell+n}_{-}$. Now, there exists a positive integer $k(a)$ for a such that

$$k > k(a) \ \text{ implies } \ f_k(a) \in \phi_k(a,\pi_k) \ \text{ a.e. in } \ A,$$

for we can take $k(a)$ as a positive integer not smaller than $\|\omega(a)\|/b$. Then $0 \le \omega^t(a)\|\omega(a)\| \le k(a)b$, $t = 1 \ldots \ell + n$.

We complete the proof by showing that

$$f(a) \in \phi(a,\pi) \ \text{ a.e in } \ A. \tag{8}$$

It follows from (7) and $f_k(a) \to f(a)$ that $\pi f(a) \le \pi\omega(a)$. Let $a \in A \backslash A(\pi)$, where $A(\pi) = \{a \in A | \ \pi(\mathbf{0},\zeta) = \pi\omega(a) \ \text{for some} \ \zeta \in \mathbb{N}^n\}$. For $\xi \in X$ with $\pi\xi < \pi\omega(a)$, we have $f(a) \succsim_a \xi$ as before. If $\pi\xi = \pi\omega(a)$, there exists a sequence ξ_i with $\xi_i \to \xi$ and $\pi\xi_i < \pi\omega(a)$, since $a \notin A(\pi)$. Then $f(a) \succsim_a \xi_i$, hence $f(a) \succsim_a \xi$ by the continuity of $\succsim_a$. Since $\nu(A(\pi)) = 0$, the claim (8) is verified and the proof is complete. ∎

Appendix II

Lemma 1. $\mathcal{E}^n(a) \to \mathcal{E}(a)$ a.e.

Proof. We show that $X^n \times X^n \to X \times X$ in the topology of closed convergence τ_c. It is clear that $Li(X^n \times X^n) \subset Ls(X^n \times X^n) \subset X \times X$. Therefore it suffices to show that $X \times X \subset Li(X^n \times X^n)$. Let $(\xi, \zeta) = ((\xi^t), (\zeta^t)) \in X \times X$, and set $\xi_n = (x^1 \dots x^n, 0, 0 \dots)$ and similarly ζ_n for ζ. Then $(\xi_n, \zeta_n) \in X^n \times X^n$ for all n and $(\xi_n, \zeta_n) \to (\xi, \zeta)$. Hence $(\xi, \zeta) \in Li(X^n \times X^n)$. Then it follows that $\succsim^n = \succsim \cap (X^n \times X^n) \to \succsim$. Obviously one obtains $\omega_n \to \omega$ in the $\sigma(\ell^\infty, \ell^1)$-topology. Consequently we have $\mathcal{E}^n(a) \to \mathcal{E}(a)$ a.e. on A. ∎

Lemma 3. $\boldsymbol{z} : A \to Z$ is Gel'fand integrable.

Proof. Since $\boldsymbol{p}\boldsymbol{z}_n(a) \to \boldsymbol{p}\boldsymbol{z}(a)$ a.e for every $\boldsymbol{p} \in \ell^1$, $\boldsymbol{z}(a)$ is weak*-measurable. Since Z is compact in the weak*-topology, it follows from the dominated convergence theorem that $\int_A |\boldsymbol{p}\boldsymbol{z}_n(a)| d\nu \to \int_A |\boldsymbol{p}\boldsymbol{z}(a)| d\nu < +\infty$ for every $\boldsymbol{p} \in \ell^1$. Hence Lemma follows fron Fact 4. ∎

Lemma 4. Let $\{f_n\}$ be a sequence of Gel'fand integrable functions from A to ℓ^∞ which converges a.e to f in the weak*-topology. Then it follows that $\int_A f_n(a) d\nu \to \int_A f(a) d\nu$ in the weak* topology.

Proof. Let $\boldsymbol{p} \in \ell^1$. Then we have

$$\boldsymbol{p} \int_A f_n(a) d\nu = \int_A \boldsymbol{p} f_n(a) d\nu \to \int_A \boldsymbol{p} f(a) d\nu = \boldsymbol{p} \int_A f(a) d\nu,$$

hence $\int_A f_n(a) d\nu \to \int_A f(a) d\nu$ in the $\sigma(\ell^\infty, \ell^1)$-topology. ∎

Lemma 7. There exists a sequence of simple functions $\{\omega_Z^n(a)\}$ with $\omega_Z^n(a) \to \omega_Z(a)$ a.e in the weak*-topology.

Proof. Since $\Omega_{idv,Z}$ is a compact metric space, it is separable. Hence, there exists a dense countable subset $\Omega_* = \{\omega_*^1, \omega_*^2 \dots\}$ of $\Omega_{idv,Z}$. For every $n \in \mathbb{N}$, let $U_n^i = \{\omega \in \Omega_{idv,Z} | \ d_Z(\omega, \omega_*^i) < 1/n\}$, where d_Z is a metric on Z. Then $\{U_n^i\}$ covers $\Omega_{idv,Z}$, or $\Omega_{idv,Z} \subset \cup_{i=1}^\infty U_n^i$ for every n. Note that U_n^i is measurable for all n, i. Define $A_n^i = \omega_Z^{-1}(U_n^i)$. Obviously A_n^i is a measurable subset of A for all n, i. Set $B_n^j = A_n^j \setminus \cup_{i=1}^{j-1} A_n^i$. Then $\cup_{j=1}^\infty B_n^j \supset A$ and $B_n^j \cap B_n^k = \emptyset$ for $j \neq k$. We then define

$$\omega_n^j(a) = \begin{cases} \omega_*^j & \text{for } a \in B_n^j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\omega_Z^n(a) = \sum_{j=1}^n \omega_n^j(a)$ is a simple function and $\omega_Z^n(a) \to \omega_Z(a)$ a.e in the weak*-topology.

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