Optimal Balanced Growth in A General Multi-sector Endogenous Growth Model with Constant Returns

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Abstract

I will study a multi-sector endogenous growth model with general constant-return-to-scale technologies and demonstrate the existence, uniqueness and the saddle-path stability of the balanced growth equilibrium. I will first show that there exists a maximum growth rate with zero consumption based on the theory developed in the von Neumann model. It follows that any balanced growth rate with positive consumption is less than that. Then I will demonstrate that the balanced growth rate is solely determined by solving a Frobenius root problem of the price equations derived from the nonsubstitution theorem.

Based on the existence of balanced growth, I will show saddle-path stability of the balanced growth equilibrium without any capital intensity conditions, which is a generalized property proved in the two-sector endogenous growth models by Bond, Wang and Yip (1996) andMino (1996). The theorem clearly implies that the balanced growth equilibrium has a transition path in the neighborhood of the balanced growth equilibrium. To show the saddle-path stability, I will exploit the properties of a flat portion of the graph of the reduced utility function, in which the balanced growth equilibrium is embedded, often referred to as "the Neumann-McKenzie facet".

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1 Introduction

Over the last twenty years, we have witnessed a resurgence of interest in Economic Growth Theory, under the titled "Endogenous Growth". These research results have been documented

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in two famous textbooks, Barro and Sala-i-Martin (1991) and Aghion and Howitt (1998). It is also nicely surveyed in Jones and Manuelli (1997). One of the main questions, among others, in Endogenous Growth Theory is what is a main driving force for generating a balanced growth path. Romer (1986) emphasized increasing returns to scale technologies that result from Marshallian externalities. On the other hand, Rebelo (1991) has claimed that linear technologies, often referred to as the "A-K technology" is a driving force of a sustainable growth of per-capita output. The asymptotic properties of the A-K technology have been studied in detail by Kaganovich (1997) under a general model setting. In contrast to the A-K models, Lucas (1988) has demonstrated that without externalities, under assumptions of constant-return-to-scale Cobb-Douglass technologies and two different kinds of reproducible capital goods (physical capital and human capital), there exists a balanced growth path. Also he analyzed its local properties. The importance of contributions by Lucas (1988) should never be underestimated, because Lucas (1988) contributed the first paper where it was shown that for sustainable growth, neither A-K technologies nor externalities are needed. From this point of view, the Lucas model is a very useful benchmark model, since it carries many common features of "Ramsey models" studied intensively in Optimal Growth Theory in 60s and 70s; for example by Uzawa (1965). Furthermore, in his model private optimality and social optimality coincide due to lack of externalities. In a series of empirical research recently, Basu and Fernald (1995, 1997) reported that returns to scale in the US economy is at most constant. That is an understandable result since in the very long-run, any firm can adjust all inputs including entrepreneurial factor as emphasized by McKenzie (1959). Since then, Mulligan and Sala-i-Martin (1993), Bond, Wang and Yip (1996) and Mino (1996) have intensively analyzed the Lucas two-sector model with constant-return-to-scale technologies and have demonstrated the existence, uniqueness and saddle-path stability of the balanced growth path. Through their demonstrations, they have fully used the duality method. Dolmas (1996) has studied a similar problem in a general Ramsey model setting and has given sufficient conditions for the existence of an endogenously growing optimal path by applying a similar method used for demonstrating the existence of an optimal steady state by McKenzie (1986,2003).

The objective of the paper is to extend these results of a two-capital goods model into those of a heterogeneous capital goods model. There are several important aspects of my extension. First, it is important to know whether or not, the results obtained in the two-sector models could be carried over to a heterogeneous capital goods model. Second, assuming general constant-return-to-scale production functions, I will explicitly calculate the balanced growth rate which is determined inside of the model. Third, all the properties used in my proofs have been well-established in the old growth theory: 1) the existence of a maximum growth rate in the von Neumann model, 2) the nonsubstitution theorem and 3) the Frobenius-Perron theorem for non-negative matrices. I will properly combine them to reach the conclusions. Also, clear analytical difference from the existing literature is that exploiting the discrete-time model structure, I will show the existence of a flat piece of the graph of the reduced form utility function, in which the balanced growth equilibrium is embedded. This flat is often referred to as the "Neumann-McKenzie Facet". By demonstrating that any path on the Neumann-McKenzie facet is explosive, I will show saddle-path stability; there exists unique stable manifold near the optimal balanced growth path or, in other words, there exists unique transition path near the optimal balanced growth path.

The structure of the paper is the following: In Section 1, I explain the model and assumptions, and show that there exists a maximum balanced growth rate with zero consumption. In Section 2, I will transform the original model into a normalized model by dividing all the variable by a certain positive growth rate. For such a model, I will define a reduced form model used in McKenzie (1986,2003). And I will also show that there exists an optimal steady state path in the reduced model, which is actually a balanced growth path in the original non-transformed model. In Section 4, the Neumann=McKenzie facet will be defined. I will show the saddle-path stability in Section 5. The final section is left for some comments.

2 The Model and Assumptions

My model is a generalized discrete-time version of the one studied by Mulligan and Sala-i-Martin (1993), Bond, Wang and Yip (1996), and Mino (1996), where twosector endogenous models with physical capital and human capital are considered. I will extend their models to an endogenous growth model with heterogeneous capital goods. A continuous-time endogenous model with heterogeneous capital goods has been studied by Benhabib, Meng and Nishimura (2000). Therefore, the model can be regarded as a discrete-time version of their model with general neoclassical production functions. The model is as follows:

$$\max imize \sum_{t=0}^{\infty} \frac{1}{(1+\eta)^t} u(C(t))$$

$$subject\ \mathbf{K}(0) = \overline{\mathbf{K}}$$

$$Y_1(t) + K_1(t) - \delta_1 K_1(t) - K_1(t+1) = C(t), \tag{1}$$

$$Y_i(t) + K_i(t) - \delta_i K_i(t) - K_i(t+1) = 0, (i = 2, ..., n) \tag{2}$$

$$Y_i(t) = f^i(K_{1i}(t), K_{2i}(t), \cdots, K_{ni}(t)), \ (i = 1, ..., n), \tag{3}$$

$$\sum_{j=0}^{n} K_{ij}(t) = K_i(t), \tag{4}$$

where $t = 0, 1, 2, \ldots$, and the notation is as follows:

$\eta \in (0, 1)$ = *subjective discount rate,*

$C(t) \in R_+$ = *per capita consumption goods consumed at t,*

$u : R_+ \longmapsto R_+$ = *representative consmer's utility function,*

$Y_i(t) \in R_+$ = *t^{th} period i^{th} per capita capital good output,*

$K_i(t) \in R_+$ = *t^{th} period i^{th} per capita capital stock,*

$\mathbf{K}(0) \in R_+^n$ = *initial period per capita capital stock vector,*

$f^j : R_+^{n+1} \longmapsto R_+$ = *per capita production function of the j^{th} sector which is strictly quasi concave, homogeneous of degree one and continuously differentiable on the interior of R_+^{n+1},*

$K_{ij}(t)$ = *i^{th} per capita capital good used in the j^{th} sector in the t^{th} period,*

δ_i = *depreciation rate of the i^{th} capital good, given as* $0 < \delta_i < 1$.

It seems meaningless to classify capital goods into human capital and the other physical capital goods under our model setting. However, I can easily set up a relationship between human capital growth models studied by other researchers and me. Let me assume that the second sector is the human capital sector and that each sector uses the human capital as much as $L_i K_{2i}$ $(i = 1, \ldots, n)$ where L_i is the labor input used in the $i - th$ sector and $\sum^n L_i = 1$. Then the production function of the $i - th$ sector can be written as:

$$Y_i(t) = f^i(K_{1i}(t), L_i(t)K_{2i}(t), K_{3i}(t), \ldots, K_{ni}(t)), \ (i = 1, \ldots, n).$$

Divide both sides by $L_i(t)$ and by using the same notation, I can obtain the per capita production functions denoted by Eq.(3). Each variable, however, is now measured by per capita.

I maintain the following standard assumptions throughout the paper.

Assumption 1. 1)The utility function $u(\cdot)$ is defined on R_+ as the following:

$$u(C(t)) = \frac{C^{\tau}(t)}{\tau}, \tau \in (0, 1).$$

This type of the utility function is often used in Macroeconomics.

Assumption 2. 1) All the goods are produced nonjointly with production functions f_i $(i = 1, \cdots, n)$ which are defined on R_+^n, homogeneous of degree one, strictly quasi-concave and continuously differentiable for positive inputs. 2) Any good j $(j = 1, \cdots, n)$ cannot be produced unless $K_{ij} > 0$

for some $i = 1, \cdots, n$. 3) The Inada-type boundary conditions hold.

I may now prove the following basic property.

Lemma 1 *Under Assumption 2, Eqs. (2)-(5) are summarized as the CRS (constant returns to scale) social production function* $Y_1(t) = T(\widetilde{\mathbf{Y}}(t), \mathbf{K}(t))$ *which is continuously differentiable on the interior* $\boldsymbol{R}_+^{2n-1}$ *and concave where*

$$\widetilde{\mathbf{Y}}(t) = (Y_2(t), Y_3(t), \cdots, Y_n(t)) \text{ and } \mathbf{K}(t) = (K_1(t), K_2(t), \cdots, K_n(t)).$$

Proof. See Benhabib and Nishimura (1979). ■

From now on I will take logic similor to that used by Drugeon et al.(2003), where they first find the maximum growth rate with zero consumption and with this growth rate they make the assumption that guarantees the infinite-summability of the discounted utility. Of course, my model contains n different capital goods and their logic cannot be directly applicable. Instead of that, I apply the well-established theorem proved in the von Neumann model which is well-documented in Takayama (1985).

Due to Lemma 1, I can define the following feasible set :

Definition 2

$$D = \{(\mathbf{X}, \mathbf{Z}) \in \mathrm{R}_+^n \times \mathrm{R}_+^n : T[\widetilde{\mathbf{Z}} - (\widetilde{\mathbf{I}} - \widetilde{\Delta})\widetilde{\mathbf{X}}, \mathbf{X}] + (1 - \delta_1)X_1 - Z_1 \geq 0\}$$

where $\mathbf{X} = \mathbf{K}(t) = (X_1, \widetilde{\mathbf{X}})$, $\mathbf{Z} = \mathbf{K}(t+1) = (Z_1, \widetilde{\mathbf{Z}})$, $\widetilde{\Delta}$ *is a following* $(n-1) \times (n-1)$*- diagonal matrix*

$$\widetilde{\Delta} = \begin{pmatrix} \delta_2 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix}$$

and $\widetilde{\mathbf{I}}$ *is an* $(n-1)$ *-dimensional unit matrix.*

Due to the homogeneity assumption of each sector's production function, it is often convenient to express a chosen technology as a technology matrix. Let me define the technology matrix as follows:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

where $a_{ij} = K_{ij}/Y_j$ $(i = 1, \cdots, n; j = 1, \cdots, \mathrm{n})$.

I make first the following two assumptions in terms of the technology matrix to show the existence of a maximum expansion rate.

Assumption 3. Any chosen technology matrix $\mathbf{A}$ is indecomposable.

Assumption 4. (*Viability*) Any chosen technology matrix $\mathbf{A}$ must satisfy the following condition:

$$[\mathbf{I}-\Delta\mathbf{A}]^{-1} \gg \Theta.$$

where Θ is a $n \times n$ zero matrix[1] and there exists a technology matrix $\overline{\mathbf{A}}$ which satisfies the above.

Assumption 3 means that any chosen technology could produce all goods so that they are more than a reproducible production level and in fact there exists such a technology in the production set D.

I am ready to prove the following lemma.

Lemma 3 *Under Assumptions 2 through 4, the production set* D *satisfies the following properties: 1) D is a closed convex cone, 2)* $(\mathbf{0}, \mathbf{Z}) \in \mathrm{D}$ *implies* $\mathbf{Z}=\mathbf{0}$, *3)* $(\mathbf{X}, \mathbf{Z}) \in \mathrm{D}$, $\mathbf{X}' \geq \mathbf{X}$ *and* $\mathbf{0} \leq \mathbf{Z}' \leq \mathbf{Z}$ *imply* $(\mathbf{X}', \mathbf{Z}') \in \mathrm{D}$, *4)There exists an* $(\mathbf{X}'', \mathbf{Z}'') \in \mathrm{D}$ *such that* $\mathbf{Z}'' \gg 0$.

Proof. Due to the fact that the social production function satisfies concavity and the constant returns to scale, I may directly prove 1) through 3). Let $\overline{\mathbf{A}}$ satisfy Assumption 4. From Assumption 4, there is an output vector $\overline{\mathbf{Y}} > 0$ such that $[\mathbf{I}-\Delta\overline{\mathbf{A}}]\overline{\mathbf{Y}} \gg \mathbf{0}$ holds. By a scalar multiplication of $\overline{\mathbf{Y}}$, I can construct $\mathbf{Y}''$so that $\mathbf{X}''=\overline{\mathbf{A}}\mathbf{Y}''$. On the other hand, from the accumulation equations,

$$\mathbf{Z}''=[\mathbf{I}+\overline{\mathbf{A}}-\Delta\overline{\mathbf{A}}]\mathbf{Y}'' > [\mathbf{I}-\Delta\overline{\mathbf{A}}]\mathbf{Y}'' \gg \mathbf{0}.$$

The last inequality comes from Assumption 4. So condition 4) holds. ■

Due to Lemma 2, I can prove the following famous proposition on the existence of a maximum expansion rate originated in the arguments raised in the von Neumann model.

Lemma 4 *There exists an* $(\overline{\mathbf{X}},\overline{\mathbf{Z}}) \in \mathrm{D}$ *such that* $\overline{\mathbf{Z}} = (1+\bar{\lambda})\overline{\mathbf{X}}$, *where* $\bar{\lambda}(>0)$ *is a maximum expansion rate with zero-consumption.*

Proof. See Theorem 6.A.1 in Takayama (1985). Assumption 3 may guarantee that $\bar{\lambda} > 0$. ■

Remark 1 *It is important to note that any balanced growth rate* λ *with non-negative consumption should satisfy* $\lambda \in [0, \bar{\lambda}]$.

Due to the above proposition, I will add the following assumption related to the maximum

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[1]Let $\mathbf{A}$ and Θ be a n-dimentional square matrix and n-dimensional zero matrix. Then $\mathbf{A} \gg \Theta$ if $a_{ij} > 0$ for all i, j, $\mathbf{A} > \Theta$ if $a_{ij} \geq 0$ for all i, j and $a_{ij} > 0$ for some i, j and $\mathbf{A} \geq \Theta$ if $a_{ij} \geq 0$ for all i, j. Also two vectors x and y in $\mathbf{R}^n$, $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all i, $\mathbf{x} > \mathbf{y}$ means $x_i \geq y_i$ for all i and with strict inequality for at least one i, and $\mathbf{x} \gg \mathbf{y}$ means $x_i > y_i$ for all i.

expansion rate $\bar{\lambda}$.

Assumption 5. (*Block-Gale Condition*)

$$\frac{(1+\bar{\lambda})^{\tau}}{(1+\eta)} < 1.$$

This assumption takes an important role in the following section. The balanced growth rate λ with non-negative consumption will satisfy that $\lambda \leq \bar{\lambda}$; thus it will automatically follow that Assumption 4 always holds for such a balanced growth rate.

3 Optimal Balanced Growth

Since I assume constant returns, I can normalize the model explained in the previous section by dividing all the variables by $(1+\lambda)^t$ where $\lambda \in (0, \bar{\lambda})$ and t denotes time. Let me define the new variables as follows:

Definition 5

$$c(t) = C(t)(1+\lambda)^{-t}, k_{ij}(t) = K_{ij}(t)(1+\lambda)^{-t} \ (i,j = 1,\cdots,n)$$

and

$$y_i(t) = Y_i(t)(1+\lambda)^{-t} \ (i = 1,\cdots,n), \ \mathbf{k}(\mathbf{0}) = \mathbf{K}(\mathbf{0}) \ (1+\lambda)^{-t}$$

Then I can rewrite the original model with new variables defined above and call it the "λ– *transformed model*":

The λ–Transformed Model:

$$Maximize \sum_{t=0}^{\infty} [(1+\lambda)^{\tau}/(1+\eta)]^t (c^{\tau}(t)/\tau) = \sum_{t=0}^{\infty} \rho^t u(c(t))$$

$$subject\ to: \mathbf{k}(0) = \bar{\mathbf{k}},$$

$$y_i(t) = f^i(k_{1i}(t), \ldots, k_{ni}(t)) \quad (i = 1, \ldots, n; t = 0, 1, \ldots),$$

$$\frac{1}{(1+\lambda)}(y_1(t) + k_1(t)(1-\delta_1)) = \frac{1}{(1+\lambda)}c(t) + k_1(1+t) \quad (t = 0, 1, \ldots),$$

$$\frac{1}{(1+\lambda)}(y_i(t) + k_i(t)(1-\delta_i)) = k_i(1+t) \quad (i = 2, \ldots, n; t = 0, 1, \ldots),$$

$$\sum_{j=0}^{\infty} k_{ij}(t) = k_i(t) \quad (i = 1, \ldots, n; t = 0, 1, \ldots).$$

Remark 2 *Let me consider λ and ρ as a rate of population growth and a subjective discount* *rate respectively. Then the λ—transformed model could be regarded as a standard neoclassical multi-sector optimal growth model. Therefore I can apply a similar method to the model as the one used in the neoclassical multi-sector optimal growth model where λ is given as a rate of population growth. However, note that the crucial difference between the endogenous growth model considered here and the standard neoclassical optimal growth model is that the growth rate λ would not be given outside the model, but should be determined inside of the model.*

Keeping this remark in mind, I can derive the following reduced form model that has also been demonstrated in the standard neoclassical optimal growth model. If $\mathbf{x}$ and $\mathbf{z}$ stand for initial and terminal capital stock vectors respectively, then the reduced form utility function $V(\mathbf{x}, \mathbf{z})$ and the feasible set D can be defined as follows:

$$V(\mathbf{x},\mathbf{z}) = u\left\{T[(1+\lambda)\tilde{\mathbf{z}} - (\tilde{\mathbf{I}} - \tilde{\Delta})\tilde{\mathbf{x}}, \mathbf{x}] + (1-\delta_1)k_1(t) - (1+\lambda)k_1(1+t)\right\}$$

and

$$D = \{(\mathbf{x},\mathbf{z}) \in R_+^n \times R_+^n : T[(1+\lambda)\tilde{\mathbf{z}} - (\tilde{\mathbf{I}} - \tilde{\Delta})\tilde{\mathbf{x}}, \mathbf{x}] + (1-\delta_1)k_1(t) - (1+\lambda)k_1(1+t)\}$$

where $\tilde{\mathbf{x}} = (k_2(t), k_3(t), \cdots, k_n(t))$, $\mathbf{x} = (k_1(t), \tilde{\mathbf{x}})$, $\tilde{\mathbf{z}} = (k_2(t+1), k_3(t+1), \cdots, k_n(t+1))$,
and $\mathbf{z} = (k_1(1+t), \tilde{\mathbf{z}})$.

Thus the above optimization problem can be summarized as the following standard reduced form problem, which is a familiar feature of the Turnpike Theory:

The Reduced-form Model:

$$\max imize \sum_{t=0}^{\infty} \rho^t V(\mathbf{k}(t), \mathbf{k}(t+1))$$

$$subject\ to\ (\mathbf{k}(t), \mathbf{k}(t+1)) \in int\,D\ \ and\ \ \mathbf{k}(0) = \bar{\mathbf{k}}\ (t = 0, 1, ...).$$

Also note that any interior optimal path must satisfy the following Euler equations, which show an intertemporal effciency allocation:

$$\mathbf{V}_z(\mathbf{k}(t-1), \mathbf{k}(t)) + \rho \mathbf{V}_x(\mathbf{k}(t), \mathbf{k}(t+1)) = \mathbf{0}\ (t \geq 0)$$

where the partial derivative vectors mean that

$$\mathbf{V}_x(\mathbf{k}(t), \mathbf{k}(t+1)) = [\partial V(\mathbf{k}(t), \mathbf{k}(t+1)) / \partial k_1(t), \ldots, \partial V(\mathbf{k}(t), \mathbf{k}(t+1)) / \partial k_n(t)]',$$
$$\mathbf{V}_z(\mathbf{k}(t-1), \mathrm{k}(t)) = [\partial V(\mathbf{k}(t-1), \mathbf{k}(t)) / \partial k_1(t), \ldots, \partial V(\mathbf{k}(t), \mathbf{k}(t-1)) / \partial k_n(t)]',$$

and $\mathbf{0}$ means an n-dimensional zero column vector. Note that under the differentiability assumptions, all the price vectors will satisfy the following relations:

$$q = \partial u(c)/\partial c,$$
$$p_i = -q\partial T(\tilde{\mathbf{y}}, \mathbf{k})/\partial y_i \ (i = 2, \cdots, n),$$

and

$$w_i = q\partial T(\tilde{\mathbf{y}}, \mathbf{k})/\partial k_i \ (i = 1, 2, \cdots, n).$$

Using these relations, I can define the price vectors of capital goods as $(n \times 1)$ column vector $\mathbf{p} = (q, p_2, \cdots, p_n)' = (q, \tilde{\mathbf{p}})$, define the output of capital goods as n dimensional column vector $\mathbf{y} = (y_1, y_2, \cdots, y_n)'$, define the rental rate as n dimensional row vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ and define the capital stock as n dimensional column vector $\mathbf{k} = (k_1, k_2, \cdots, k_n)'$, where " $'$ " implies transposition of vectors. From now on, I will divide all the prices by q and keep the same notation for these normalized vectors. Thus $\mathbf{p} = (1, p_2, \cdots, p_n)' = (1, \tilde{\mathbf{p}})$.

Definition 6 *An optimal balanced growth path $\mathbf{K}^*(t)$ (denoted by OBG henceforth) is an optimal path which solves the original optimization problem [Eq. (1)-(4)] and $\mathbf{K}^*(t) = (1+\lambda)^t\mathbf{k}^\lambda$ for all $t \geqslant 0$ and a given $\mathbf{k}^\lambda \gg \mathbf{0}$.*

To emphasize that the growth rate λ is determined inside of the model, from now on I will put the symbol λ on each variables as a superscript.

Remark 3 *It is important to note that the OBG defined above is equivalent to an optimal steady state $\mathbf{k}^\lambda$ (denoted by OSS) which solves the λ-normalized model and $\mathbf{k}^\lambda = \mathbf{k}(t) = \mathbf{k}(t+1)$ for all $t \geqslant 0$. In fact, multiplying by $(1+\lambda)^t$ on both sides of $\mathbf{k}^\lambda = \mathbf{k}(t)$ results in $(1+\lambda)^t\mathbf{k}^\lambda = (1+\lambda)^t\mathbf{k}(t)$. Thus it follows that $\mathbf{K}(t) = (1+\lambda)^t\mathbf{k}^\lambda$ for all $t \geqslant \mathbf{0}$.*

McKenzie (1986,2002) has shown the existence of an optimal path and the OSS in the reduced form model. Dolmas (1996) has applied the McKenzie's method to prove the existence of balanced growth in a general Ramsey model. Instead of that, I will take a more direct method to show the existence of the OBG. In other words, I will directly derive the OGB from my model setting.

Suppose that $\mathbf{k}^\lambda \gg \mathbf{0}$ is an interior OSS with a given growth rate $\lambda > 0$, then it must satisfy the Euler equations:

$$\mathbf{V}_z(\mathbf{k}^\lambda, \mathbf{k}^\lambda) + \rho\mathbf{V}_x(\mathbf{k}^\lambda, \mathbf{k}^\lambda) = 0 \tag{5}$$

noting again that

$$\rho = (1+\lambda)^\tau/(1+\eta).$$

Due to the above definition of OSS, I will express the partial derivatives of the Euler equations in terms of price vectors:

$$\mathbf{V}_x(\mathbf{k}^\lambda, \mathbf{k}^\lambda) = \mathbf{p}^\lambda(\mathbf{I} - \Delta) + \mathbf{w}^\lambda,$$
$$\mathbf{V}_z(\mathbf{k}^\lambda, \mathbf{k}^\lambda) = -(1+\lambda)\,\mathbf{p}^\lambda.$$

where $\mathbf{I}$ is an $n \times n$ unit matrix. Substituting these relations into the Euler equations may yield the following:

$$\rho[\mathbf{w}^\lambda + \mathbf{p}^\lambda(\mathbf{I} - \Delta)] - (1+\lambda)\,\mathbf{p}^\lambda = 0.$$

Since a subjective discount rate η is given, I can derive that $(1+\lambda)/\rho = (1+\eta)/(1+\lambda)^{\tau-1} = 1 + g(\lambda)$ where the last equality comes from Assumption 4 and the function $g(\lambda)(>0)$ is a continuous function of λ. Also note that since $g'(\lambda) = (1+\eta)(1-\tau)(1+\lambda)^{-\tau} > 0$ *for* $\tau \in (0, 1)$, the inverse function of $g(\lambda)$ is definable. Then urther calculation will finally yield:

$$[g(\lambda)\mathbf{I} + \Delta]\mathbf{p}^\lambda = \mathbf{w}^\lambda. \tag{6}$$

or

$$p_i^\lambda = \frac{w_i^\lambda}{g(\lambda) + \delta_i}\ (i = 1, ..., n)$$

These equations mean non-arbitrage conditions among capital goods and imply that any capital good must yield the same rate of return as the rate $g(\lambda)$. Thus the Euler conditions are the non-arbitrage conditions. Let me denote the unit-cost function by $C^i(\mathbf{w})$ $(i = 1, \cdots, n)$, which is homogeneous of degree one and $\partial C^i/\partial w_j = a_{ij}$ where $a_{ij} = k_{ij}/y_j$ $(i = 1, 2, \cdots, n;\ j = 1, \cdots, n)$. Due to the cost minimization condition, I can have the following equations:

$$\frac{w_i^\lambda}{g(\lambda) + \delta_i} = C^i(w_1^\lambda, ..., w_n^\lambda)\ (i = 1, ..., n)$$

Because of the differentiability and constant returns to scale technologies, the following well-known proposition, referred to as the Nonsubstitution Theorem originally proved by Samuelson (1945) and generalized to a multi-sector neoclassical growth model by Burmeister and Kuga (1970), and Burmeister and Dobell (1970), will hold:

Proposition 7 *The normalized rental rate* $w_i^\lambda (i = 1, ..., n)$ *is a continuous function of* $\lambda \in [0, \bar{\lambda}]$.

Proof. See theorem 2 of Burmeister and Kuga (1970) and Section 9.4 in Burmeister and Dobell (1970). ■

The theorem means that all the real rental rates are determined independently of the composition of output, but solely by the growth rate. So I can treat the quantity and the price systems independently. Due to this theorem, all the prices p_i $(i = 1, ..., n)$ and a technology matrix **A** are also expressed as continuous functions of $\lambda \in [0, \bar{\lambda}]$.

On the other hand, the cost-minimization condition can be expressed by the technology matrix **A** as follows:

$$\mathbf{p}^{\lambda} = \mathbf{w}^{\lambda}\mathbf{A}^{\lambda}.$$

Substituting this relation into the previous equation (6) yields

$$\{\mathbf{I} - [g(\lambda)\mathbf{I} + \Delta]\mathbf{A}^{\lambda}\}\mathbf{w}^{\lambda} = \mathbf{0}. \tag{7}$$

To emphasize the continuity due to the Nonsubstitution Theorem, I will express vectors $\mathbf{A}^{\lambda}$ and $\mathbf{w}^{\lambda}$ as $\mathbf{A}(\lambda)$ and $\mathbf{w}(\lambda)$ respectively.

Now I will demonstrate that by exploiting these relations, there exists a growth rate $\lambda^* \in [0, \bar{\lambda}]$ such that the following equation holds:

$$\{\mathbf{I} - [g(\lambda^*)\mathbf{I} + \Delta]\mathbf{A}(\lambda^*)\}\mathbf{w}(\lambda^*) = \mathbf{0}.$$

Note that the above equation can be rewritten as:

$$\begin{aligned} &[\mathbf{I} - \Delta\mathbf{A}(\lambda^*)]^{-1}\{\mathbf{I} - g(\lambda^*)[\mathbf{I} - \Delta\mathbf{A}(\lambda^*)]^{-1}\mathbf{A}(\lambda^*)\}\mathbf{w}(\lambda^*) \\ &= [\mathbf{I} - \Delta\mathbf{A}(\lambda^*)]^{-1}[\mathbf{I} - g(\lambda^*)\mathbf{E}(\lambda^*)]\mathbf{w}(\lambda^*) = \mathbf{0} \end{aligned}$$

From Assumption 3, $[\mathbf{I} - \Delta\mathbf{A}(\lambda^*)]^{-1} \gg \Theta$ and the matrix $\mathbf{A}(\lambda^*)$ is indecomposable, therefore $\mathbf{E}(\lambda^*) = \mathbf{I} - \Delta\mathbf{A}(\lambda^*)]^{-1}\Delta\mathbf{A}(\lambda^*)$ is a positive matrix and thus it is also indecomposable. Now I can apply the proof used in the Frobenius root and demonstrate that there exists a growth rate $\lambda^* > 0$ which is calculated from the Frobenius root of the matrix $\mathbf{E}(\lambda^*)$.

Before that I need to provide some definitions:

Definition 8 *For every positive vector* **x**,

$$r_{\mathbf{x}} = \min_{x_i \neq 0} \frac{(\mathbf{Ex})_i}{x_i}.$$

Definition 9 $\bar{\mathbf{S}} = \{\mathbf{y} : \mathbf{y} = (\mathbf{I} + \mathbf{E})^{n-1}\mathbf{x}$, *where* $\mathbf{x}\mathbf{x}' = \mathbf{1}\}$

Then I can demonstrate the following lemma.

Lemma 10 *If* **E** *is an indecomposable matrix, its Frobenius root will be obtained by solving the following maximization problem:*

$$\underset{\mathbf{y} \in \bar{\mathbf{S}}}{Max}\ r_y.$$

Proof. See pp. 372-375 in Takayama (1977) and pp. 27-28 in Berma and Plemmmons (1979). ■

Finally I can demonstrate the following important property:

Corollary 11 *There exists a maximum growth rate* $\lambda^* > 0$ *which satisfies*

$$\{\mathbf{I} - [\, g\,(\lambda^*)\mathbf{I} + \Delta]\,\mathbf{A}(\lambda^*)\}\,\mathbf{w}(\lambda^*) = \mathbf{0}.$$

Proof. Since $g(\lambda)$ has an inverse function g^{-1} and $g'(\lambda) > 0$, let me define the following function:

$$\Gamma(\lambda) \equiv g^{-1}\left\{ 1 = \underset{\mathbf{y}\in\overline{\mathbf{S}}}{Max}\left(\min_{y_i \neq 0} \frac{(E(\lambda)\mathbf{y})_i}{y_i} \right) \right\} \; where \; \lambda \in [0, \overline{\lambda}].$$

Since $\mathbf{E}(\lambda)$ is an indecomposable matrix for a given $\lambda \in [0, \overline{\lambda}]$, from the Frobenius theorem, $\Gamma(\lambda)$ satisfies $0 < \Gamma(\lambda) < \lambda$. Therefore $\Gamma(\lambda)$ is a single-valued continuous function from $[0, \overline{\lambda}]$ into itself. Due to the Brouwer's fixed point theorem, there exists a λ^* such that $\lambda^* = \Gamma(\lambda^*)$ and a corresponding eigenvector $\mathbf{z}^* \gg \mathbf{0}$. In other words, $\{\mathbf{I} - [g(\lambda^*)\mathbf{I} + \Delta]\mathbf{A}(\lambda^*)\}\mathbf{z}^* = \mathbf{0}$. Since $1/g(\lambda^*)$ is the Frobenius root and $\mathbf{z}^*$ is a unique corresponding eigenvector, thus I can put $\mathbf{z}^* = \mathbf{w}(\lambda^*)$. This completes the proof. ■

From the theorem, the growth rate λ^* is determined within the model. Then all the price vectors are also determined by the growth rate λ^*. Thus by using these price vectors in the λ−transformed model, unique output vector $\mathbf{y}^* \gg \mathbf{0}$ and capital stock vector $\mathbf{k}^* \gg \mathbf{0}$ will be determined through the social production function $T(\tilde{\mathbf{y}}, \mathrm{k})$.

Thus, by defining $\mathbf{K}^*(t) = (1+\lambda^*)^t\mathbf{k}^*$ as the BG path, I have finally proved the following corollary of the theorem:

Theorem 12 *Under my assumptions, there exist a unique optimal balanced growth path* $\mathbf{K}^*(t)$ *with the growth rate* $\lambda^* > 0$; $\mathbf{K}^*(t) = (1+\lambda^*)^t\mathbf{k}^*$ *for all* $t \geqslant 0$.[2]

Proof. I will demonstrate first that $\mathbf{K}^*(t) = (1+\lambda^*)^t\mathbf{k}^*$ is an optimal balanced growth path. Since $\mathbf{K}^*(t)$ has a maximum balanced growth rate, the corresponding consumption path $\mathbf{C}^*(t)$ never be dominated by any other balanced growth consumption path. Note also that due to Assumption 5 (the Block and Gale Condition),

$$\begin{aligned} \lim_{t\to\infty} \rho^t p^* k^* &= \lim_{t\to\infty}\left[\frac{(1+\lambda)^{\tau}}{(1+\eta)}\right]^t p^* k^* \\ &= \lim_{t\to\infty}\left[\frac{(1+\lambda)^{\tau-1}}{(1+\eta)}\right]^t p^* k^* (1+\lambda)^t = 0 \end{aligned}$$

thus the path $\mathbf{K}^*(t)$ satishies the transverslity condition. Thus $\mathbf{K}^*(t)$ is an optimal path for the original problem. Now I will show that the optimal path is unique.

Let us suppose that there exists an another optimal consumption path $\mathbf{C}^{**}(t)$ with a maximum

[2] I have shown only the existence of a balanced growth path. However, under the given balanced growth rate, I can apply the existence theorem of the optimal path by McKenzie (1988, 2003) and show the existence of the optimal path.

growth rate λ^{**}. Now let us define the following new consumption path

$$\overline{C}(t) = \alpha C^{*}(t) + (1-\alpha)\, C^{**}(t).$$

From the following relation due to the concavity of social production function $T(\cdot)$, it is clear that the constructed consumption $\overline{C}(t)$ is feasible:

$$\begin{aligned}
&T\{[\alpha\widetilde{\mathbf{Z}}^{*} + (1-\alpha)\,\widetilde{\mathbf{Z}}^{**}] - (\widetilde{\mathbf{I}} - \widetilde{\Delta})[\alpha\widetilde{\mathbf{X}}^{*} + (1-\alpha)\,\widetilde{\mathbf{X}}^{**}], [\alpha\mathbf{X}^{*} + (1-\alpha)\mathbf{X}^{**}]\}\\
&+ (1-\delta_1)[\alpha\mathbf{X}_1^{*} + (1-\alpha)\mathbf{X}_1^{**}] - [\alpha\mathbf{Z}_1^{*} + (1-\alpha)\mathbf{Z}_1^{**}]\\
&\geq \{T[\alpha\widetilde{\mathbf{Z}}^{*} - (\widetilde{\mathbf{I}} - \widetilde{\Delta})\,\alpha\widetilde{\mathbf{X}}^{*}, \alpha\mathbf{X}^{*}] + (1-\delta_1)\,\alpha\mathbf{X}_1^{*} - \alpha\mathbf{Z}_1^{*}\} + \{T[(1-\alpha)\,\widetilde{\mathbf{Z}}^{**}]\\
&- (\widetilde{\mathbf{I}} - \widetilde{\Delta})(1-\alpha)\,\widetilde{\mathbf{X}}^{**}, (1-\alpha)\mathbf{X}^{**}] + (1-\delta_1)(1-\alpha)\mathbf{X}_1^{**} - (1-\alpha)\mathbf{Z}_1^{**}\}\\
&= \alpha C^{*}(t) + (1-\alpha)C^{**}(t) = \overline{C}(t).
\end{aligned}$$

Then from the strict concavity of the utility function $u(\mathrm{C}(t))$ and summability of the original problem, it follows that

$$\begin{aligned}
\sum_{t=0}^{\infty}\frac{1}{(1+\eta)^{t}}u[\alpha C^{*}(t)+(1-\alpha)C^{**}(t)] &= \alpha\sum_{t=0}^{\infty}\frac{1}{(1+\eta)^{t}}u[\overline{C}(t)]\\
&> \alpha\sum_{t=0}^{\infty}\frac{1}{(1+\eta)^{t}}u[C^{*}(t)]\\
&\quad + (1-\alpha)\sum_{t=0}^{\infty}\frac{1}{(1+\eta)^{t}}u[C^{**}(t)]
\end{aligned}$$

This is a contradiction to the fact that the paths $\{C^{*}(t)\}$ and $\{C^{**}(t)\}$ are optimal. Thus $C^{*}(t) = C^{**}(t)$. This completes the proof. ■

4 The Neumann-McKenzie Facet

Now I will introduce the Neumann-McKenzie Facet (NMF for short), which takes important roles in stability arguments of neoclassical growth models as studied in Takahashi (1985) and has been intensively studied by McKenzie [see especially McKenzie (1986)]. The NMF can be defined for the reduced form growth model as follows:

Definition 13 *The Neumann-McKenzie Facet of the OSS* $\mathbf{k}^{*}$*, denoted as* $\mathbf{F}(\mathbf{k}^{*}, \mathbf{k}^{*})$*, is defined as:*

$$\mathbf{F}(\mathbf{k}^{*}, \mathbf{k}^{*}) = \{(\mathbf{x}, \mathbf{z}) \in \mathrm{D} : V(\mathbf{x}, \mathbf{z}) + \rho^{*}\mathbf{p}^{*}\mathbf{z} - \mathbf{p}^{*}\mathbf{x} = V(\mathbf{k}^{*}, \mathbf{k}^{*}) + \rho^{*}\mathbf{p}^{*}\mathbf{k}^{*} - \mathbf{p}^{*}\mathbf{k}^{*}\}$$

where $\mathbf{k}^{*}$ *is the OSS with the growth rate* λ^{*} *and* $\mathbf{p}^{*}$ *is a supporting price of* $\mathbf{k}^{*}$ *where* $\rho^{*} = (1+\lambda^{*})^{\tau}/(1+\eta)$.

From the definition above, the NMF is the projection of a flat piece of the graph of the function

V that is supported by the price vector $(-\mathbf{p}^*, \rho^*\mathbf{p}^*, 1)$ onto the $(\mathbf{x}, \mathbf{z})$ space. In Takahashi (1985), I considered the case of the objective function where n capital goods as well as pure-consumption good are also consumable. Here, the capital goods are not consumable except the good produced in the first sector.

By exploiting this fact, I will re-characterize the NMF as a more tractable formula with the $(n \times n)$ matrix $\mathbf{A}^*$ and $(n+1)$-dimensional vectors as follows:

Lemma 14 *Suppose that* $\mathbf{A}^*$ *is non-singular chosen technology matrix along OSS* $\mathbf{k}^*$, *then* $(\mathbf{x}, \mathbf{z}) \in \mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ *if and only if* $\mathbf{y} \geq \mathbf{0}$ exists such that

$$i)\ c = c^*$$

$$ii)\ \mathbf{x} = \mathbf{A}^*\mathbf{y}$$

$$iii)\ \mathbf{z} = \frac{1}{(1+\lambda^*)}[\mathbf{y} + (\mathbf{I} - \Delta)\mathbf{x} - \mathbf{c}]$$

where $\mathbf{c} = (c, 0, ..., 0)$.

Proof. From the definition of the NMF, I have the following supporting relation:

$$u(c) + \rho^*\mathbf{p}^*\mathbf{z} - \mathbf{p}^*\mathbf{x} = u(c^*) + \rho^*\mathbf{p}^*\mathbf{k}^* - \mathbf{p}^*\mathbf{k}^*$$

where c and $\mathbf{y}$ correspond to $(\mathbf{x}, \mathbf{z})$. Furthermore, from the fact that $\mathbf{V}_x(\mathbf{k}^*, \mathbf{k}^*) = \mathbf{w}^* + \mathbf{p}^*(\mathbf{I} - \Delta)$ and $\mathbf{V}_z(\mathbf{k}^*, \mathbf{k}^*) = -(1+\lambda^*)\mathbf{p}^*$, the Euler equation implies that $[g(\lambda^*)\mathbf{I} + \Delta]\mathbf{p}^* = \mathbf{w}^*$. Note that $\mathbf{p} = (1, p_2, \cdots, p_n)' = (1, \tilde{\mathbf{p}})$ and $\mathbf{w} = (w_1, w_2, \cdots, w_n)$. Also note that from the accumulation relations, $\mathbf{z} = 1/(1+\lambda^*)[\mathbf{y} + (\mathbf{I} - \Delta)\mathbf{x} - \mathbf{c}^*]$ and $\mathbf{k}^\rho = 1/(1+\lambda^*)[\mathbf{y}^* + (\mathbf{I} - \Delta)\mathbf{k}^* - \mathbf{c}^*]$. Substituting theses relations into the above supporting equation, I finally obtain,

$$[u(c) - c] + [y_1 + \tilde{\mathbf{p}}^*\tilde{\mathbf{y}} - \mathbf{w}^*\mathbf{x}] = [u(c^*) - c^*] + [y_1^* + \tilde{\mathbf{p}}^*\tilde{\mathbf{y}}^* - \mathbf{w}^*\mathbf{k}^*]$$

or

$$\{[u(c) - c] - [u(c^*) - c^*]\} + \{[\mathbf{p}^*\mathbf{y} - \mathbf{w}^*\mathbf{x}] - [\mathbf{p}^*\mathbf{y}^* - \mathbf{w}^*\mathbf{k}^*]\} = 0. \quad (8)$$

Suppose $c^* \neq c$. Due to the strict concavity of the utility function u, the first term denoted by the braces of the left-hand side in the above equation will then be strictly negative. This implies that the second term should be positive in order to maintain equality. However, this contradicts the concavity of the production possibility frontier $T(\tilde{\mathbf{y}}, \mathbf{k})$. Therefore, it follows that $c = c^*$ and

$$\mathbf{p}^*\mathbf{y} - \mathbf{w}^*\mathbf{x} = \mathbf{p}^*\mathbf{y}^* - \mathbf{w}^*\mathbf{k}^*.$$

The first result is Condition $i)$. The second one implies that $(y_1, \tilde{\mathbf{y}}, \mathbf{x})$ or $(\mathbf{y}, \mathbf{x})$should lie on the production frontier of $T(\tilde{\mathbf{y}}, \mathbf{k})$ and in each sector, the technology used must be the same as that in the OSS. In other words, the OSS technology matrix $\mathbf{A}^*$ will be chosen. Thus on the NMF, the exact same technology matrix as the corresponding OSS is chosen. In other words, given OSS technology matrix $\mathbf{A}^*$, the cost minimization and the full-employment conditions for capital goods are satisfied.

Thus, the following equations must hold:

$$1)\ \mathbf{p}^* = \mathbf{w}^*\mathbf{A}^*,$$
$$2)\ \mathbf{x} = \mathbf{A}^*\mathbf{y}$$

The cost-minimization conditions 1) implies that the same technology as that of OSS is chosen. 2) means that, under the chosen technology, the full employment conditions hold. It is not difficult to see that 2) can be summarized as Condition *ii)*. From these conditions, it follows that $c(t) > 0$ and $\mathbf{y}(t) \gg \mathbf{0}$ for all t, respectively. Condition iii)are capital accumulation equations and $\mathbf{z}$ is determined through this relation. ■

It is important to notice that the dimension of the VMF could be zero. The following lemma will give an exact order of its dimension.

Lemma 15 $dim\, \mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho) = n.$

Proof. Let me define $\mathbf{d} = (d_1, \cdots, d_n)^t \in \mathbf{R}^n_+$. Since no degrees of freedom are lost, I can exactly choose n linearly independent vectors $\mathbf{d}^h$ ($h = 1, \cdots, n$). Moreover, define the following: for $h = 1, 2, \cdots, n$ and a positive scalar ε_h,

$$\begin{aligned} \mathbf{y}^h &\equiv \mathbf{y}^* + \varepsilon_h \mathbf{d}^h, \\ \mathbf{x}^h &\equiv \mathbf{A}^*\mathbf{y}^h = \mathbf{A}^*\mathbf{y}^* + \varepsilon_h \mathbf{A}^*\mathbf{d}^h \\ &= \mathbf{k}^* + \varepsilon_h \mathbf{A}^*\mathbf{d}^h \end{aligned}$$

and

$$\begin{aligned} \mathbf{z}^h &\equiv \frac{1}{(1+\lambda^*)}[\mathbf{y}^h + (\mathbf{I} - \Delta)\mathbf{x}^h - \mathbf{c}^*] \\ &= \frac{1}{(1+\lambda^*)}[\mathbf{y}^* + (\mathbf{I} - \Delta)\mathbf{x}^* - \mathbf{c}^*] + \varepsilon_h \frac{1}{(1+\lambda^*)}[\mathbf{d}^h + (\mathbf{I} - \Delta)\mathbf{A}^*\mathbf{d}^h] \\ &= \mathbf{k}^* + \varepsilon_h \frac{1}{(1+\lambda^*)}[\mathbf{I} + (\mathbf{I} - \Delta)\mathbf{A}^*]\mathbf{d}^h \end{aligned}$$

Due to the fact that $\mathbf{y}^* \gg \mathbf{0}$ and $\mathbf{k}^* \gg \mathbf{0}$, ε_h can be chosen such that $\mathbf{y}^h \gg \mathbf{0}$, $\mathbf{x}^h \gg \mathbf{0}$ and $\mathbf{z}^h \gg \mathbf{0}$ hold for all h. From this way of construction, the vectors $\mathbf{y}^h$, $\mathbf{x}^h$ and $\mathbf{z}^h$ satisfy Lemma 2 and the corresponding vector $(\mathbf{x}^h, \mathbf{z}^h)$ also belongs to $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ for all h. This implies that there are n linearly independent vectors $(\mathbf{x}^h - \mathbf{k}^*, \mathbf{z}^h - \mathbf{k}^*)$.

Therefore, there are exactly n linearly independent line segments on $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$. In other words, the NMF has an n-dimensional facet (flat) containing $(\mathbf{k}^*, \mathbf{k}^*)$. This completes the proof. ■

It is important to note the sharp contrast with the case of an exogenous growth model where one degree of freedom is lost from the whole commodity dimension because of the labor constraint. In an endogenous growth case, there are no primary input such as labor. So no degrees of freedom are lost.

Now let us consider the NMF, $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$. Note that the dynamics of the NMF is expressed by the accumulation equation of Condition *iii)* in Lemma 14. It can be rewritten by using the following equation:

$$\mathbf{y}(t) = \mathbf{B}^*\mathbf{k}(t)$$

where $\mathbf{B}^*$ is the inverse matrix of $\mathbf{A}^*$.

Combining this equation with the accumulation equation yields

$$\mathbf{k}(t+1) = \left(\frac{1}{1+\lambda^*}\right)(\mathbf{B}^* + \mathbf{I} - \Delta)\ \mathbf{k}(t) - \left(\frac{1}{1+\lambda^*}\right)\mathbf{c}^*.$$

Defining $\xi(t) = \mathbf{k}(t) - \mathbf{k}^*$,

$$\xi(t+1) = \frac{1}{1+\lambda^*}(\mathbf{B}^* + \mathbf{I} - \Delta)\xi(t) \tag{9}$$

This difference equation shows the effect of dynamic behaviors on the NMF. It is convenient to rewrite the above matrix in the equation as follows:

$$\mathbf{B}^* + \mathbf{I} - \Delta = (\mathbf{A}^*)^{-1} + \mathbf{I} - \Delta = \mathbf{I} + (\mathbf{A}^*)^{-1}[\mathbf{I} - \Delta\mathbf{A}^*] = \mathbf{I} + (\mathbf{E}^*)^{-1}.$$

Since the previous argument in Section 1, the matrix $\mathbf{E}^*$ has the Fobenius root $1/g(\lambda^*)$. Due to the Frobenius theorem, this implies that $|\alpha_i| < 1/g(\lambda^*)$ where $|\alpha_i|$ $(i = 2, ..., n)$ is an eigenvalue other than the Frobenius root and since the eigenvalues of the matrix $(\mathbf{E}^*)^{-1}$ is the inverse of the roots of the matrix $\mathbf{E}^*$. Thus $1/|\alpha_i| > g(\lambda^*)$ holds. Furthermore I have easily derived the following relation:

$$\frac{\frac{1}{|\alpha_i|}+1}{1+\lambda^*} > \frac{1+g(\lambda^*)}{1+\lambda^*} = \frac{1}{\rho^*} > 1$$

where I use the following relation to show the last equation:

$$1 + g(\lambda^*) = \frac{1+\lambda^*}{\rho^*}.$$

From this observation, I have demonstrated the following lemma:

Lemma 16 *The NMF defined above is an unstable linear manifold. In other words, the difference equation Eq.(9) has n eigenvalues with modulus greater than one.*

Proof. From my argument, the matrix $\frac{1}{1+\lambda^*}[\mathbf{I} + (\mathbf{E}^*)^{-1}]$ has the $(n-1)$ eigenvalues $\frac{\frac{1}{|\alpha_i|}+1}{1+\lambda^*} > 1$ and

$\frac{1}{\rho^*} > 1$. Thus the difference equation is explosive and the NMF is an unstable linear-manifold. ■

5 Saddle-path Stability

Since an optimal path satisfies the Euler equations Eq.(6), a linear approximation of the Euler equation around $(\mathbf{k}^*, \mathbf{k}^*)$ yields the following linear difference equation,

$$\mathbf{V}^*_{xz}\mathbf{z}(t+1) + [\mathbf{V}^*_{xx} + (\rho^*)^{-1}\mathbf{V}^*_{zz}]\,\mathbf{z}(t) + (\rho^*)^{-1}(\mathbf{V}^*_{xz})^{-1}\mathbf{V}^*_{zx}\mathbf{z}(t-1) = \mathbf{0} \tag{10}$$

where $\mathbf{z}(t) = \mathbf{k}(t) - \mathbf{k}^*$ and all the matrices are evaluated at $\mathbf{k}^*$. Furthermore the characteristic equation of Eq.(10) is the following:

$$|\,\mathbf{V}^*_{xz}\alpha^2 + [\mathbf{V}^*_{xx} + (\rho^*)^{-1}\mathbf{V}^*_{zz}]\alpha + (\rho^*)^{-1}\mathbf{V}^*_{zx}\,| = 0. \tag{11}$$

To show the saddle-path stability, I need to utilize the following well-known lemma by Levhari and Liviatan (1972):

Lemma 17 *Provided that $\alpha \neq 0$, if the characteristic equation Eq.(11) has a as a root of the equation, then it also has $1/(\rho^*\alpha)$ as its root.*

Then combining the results of Lemma 16 and Lemma 17, I will have proven the following theorem:

Theorem 18 *The balanced growth equilibrium is saddle-path stable.*

Proof. Since the optimal steady state in the λ-normalized model corresponds to the balanced growth path in the original model, it is sufficient to show that the OSS $\mathbf{k}^*$ satisfies the saddle-point stability. From Lemma 10, the NMF is an unstable linear manifold and has the $(n-1)$ eigenvalue $\beta_i = \frac{\frac{1}{|\alpha_i|}+1}{1+\lambda^*} > 1$ and $\beta_1 = \frac{1}{\rho^*} > 1$. Also $\frac{\frac{1}{|\alpha_i|}+1}{1+\lambda^*} > \frac{1}{\rho^*}$ holds. Multiplying by ρ^* on both sides of the relation yields $\rho^*\beta_i > \rho^*\beta_1 = 1$. This implies that Eq.(10) has also $(n-1)$ eigenvalues $\frac{1}{\rho^*\beta_i}$ $(i = 2, ..., n)$ and $\frac{1}{\rho^*\beta_1} = 1$. Since the original model has been normalized with the growth factor $(1+\lambda^*) > 1$, then it follows that the original model has $\frac{1}{\rho^*\beta_i(1+\lambda^*)} < 1 (i = 2, ..., n)$ and $\frac{1}{\rho^*\beta_1(1+\lambda^*)} < 1$ as its eigenvalues. Thus the balanced growth path in the original model satisfies the saddle-path stability. ■

Remark 4 *Note that Theorem 2 is a generalization of the saddle-path stability proved by Bond, Wang and Yip (1996) and Mino (1996) for a two-sector case. Also note that no capital intensity conditions are never needed which are assumed in an exogenous multi-sector growth model as Takahashi (1985).*

5.1 Concluding Remarks

I have shown that there exists a balanced growth path with a positive growth rate. Through this demonstration, the Non-substitution Theorem has taken an important role. Then the local property of the balanced growth path was also studied. I have demonstrated that a balanced growth equilibrium exhibits saddle-path stability without any capital intensity conditions. From the results, I may conclude that the existence and the saddle-path stability of the optimal balanced growth in the twosector endogenous growth model shown by Bond et al. (1996) and Mino (1996) also hold in the multi-sector endogenous growth model.

It may be an important question to ask whether or not I can show the turnpike property in the multi-sector endogenous growth model. The turnpike property means that any optimal solution path will converge to the optimal balanced growth equilibrium without depending on the choice of initial capital stocks. I have already proven the saddle-path stability, so all I need to show for the turnpike property is the "visit lemma" proved by Scheinkman (1976); any optimal path will visit the neighborhood of the balanced growth equilibrium at least once. For this property, i will usually apply the value-loss approach used by Scheinkman (1976) and McKnenzie (1986).
For demonstrating it, I have to change the discountrate η. However if I change the discount rate, then the optimal balanced growth rate λ^* will also change. So I cannot apply the standard approach for demonstrating the turnpike theorem.

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