An Unbalanced Multi-sector Growth Model with Constant Returns: A Turnpike Approach

Harutaka Takahashi*

Abstract

Recent industry-based empirical studies among countries exhibit that individual industry's per-capita capital stock and output grow along its own steady state at industry's own growth rate, which is highly correlated to industry's technical progress measured by the total factor productivity of the industry. Let us refer to this phenomenon as "unbalanced growth among industries". Since it is totally based on the highly aggregated macro-production functions, "New Growth Theory" cannot explain the unbalanced growth phenomenon. On the other hand, although Turnpike Theory is based on the multi-sector models, it demonstrates that all sectors have a common growth rate and each sector's per-capita capital stock and output converge to its own constant ratio. Therefore, Turnpike Theory cannot explain the phenomenon either.

I will set up the multi-sector optimal growth model with a sector specific Harrod neutral technical progress and show that each sector's per-capita capital stocks and outputs grow at its own rate of sector's technical progress by applying the theoretical method developed by McKenzie and Scheinkman in Turnpike Theory.

JEL Classification: O14, O21, O24, O41

1 Introduction

Since the seminal papers by Romer (1986) and Lucas (1988), we have witnessed a strong revival of interest in Growth Theory under the name of Endogenous Growth Theory, and especially, neoclassical optimal growth models have been used as analytical benchmark models, which have been intensively studied in late 60's. However, these research models have a serious drawback. Since the models are based on the highly aggregated macro-production function, they cannot explain the important empirical evidence, as I will give a detailed discussion in the following section. Recent empirical studies at the industry level among

* The paper is prepared for the International Conference on "Globalization, interdependences and macroeconomic fluctuations", held in Paris, June 11-13, 2009. The paper is very preliminay. Please do not quote.

countries provide a clear evidence that individual industry's per-capita capital stock and output grow at industry's own growth rate, which is closely related to industry's technical progress measured by the total factor productivity of the industry. For example, the per-capita capital stock and output of an agriculture sector grow at 5% per annum along its own steady state, on the other hand, those of a manufacturing sector grow at 10% per annum along its own steady state. Let us refer to this phenomenon as "unbalanced growth among industries". To tackle the problem, it has raised a strong theoretical demand for constructing a multi-sector growth model. In spite of strong needs for such a model, very little study of this type of model has been done so far.

On the other hand, the optimal growth model with heterogeneous capital goods has been intensively studied under the title of Turnpike Theory since the early 70's by McKenzie (1976, 1982, 1983 and 1986) and Scheinkman (1976). The Turnpike Theory shows that any optimal path asymptotically converges to the corresponding optimal steady state path without initial stock sensitivity. In other words, the turnpike property implies that the per-capita capital stock and output of each sector eventually converge to a sector specific constant ratio at the common growth rate. Therefore, the Turnpike Theory cannot explain the empirical phenomenon: unbalanced growth among industries, either. An additional point to notice is that the Turnpike result established in the reduced form model, which has not been fully applied to a structure model: a neoclassical optimal growth model. One serious obstacle to apply the results given in the reduced model is that transforming a neoclassical optimal growth model into the reduced model will not yield a strictly concave reduced form utility function, but just a concave one. In this context, McKenzie (1983) has extended the Turnpike property to the case in which the reduced form utility function is not strictly concave: that is, it has a flat on the surface in which an optimal steady state is contained. This flat is often referred to as the Neumann-McKenzie facet.Yano (1990) has studied a neoclassical optimal growth model with heterogeneous capital goods in a trade theoretic context. However, in the case of the Neumann-Mckenzie facet with a positive dimension, Yano made a direct assumption called the dominant diagonal block condition concerned with the reduced form utility function (see Araujo and Scheinkman (1978) and McKenzie (1986)). Thus he still did not fully exploit the structure of the neoclassical optimal growth model, especially the dynamics of the path on the Neumann-McKenzie facet to obtain the Turnpike property.

The paper is undertaken to fill the gap between the results derived by the theoretical researches explained above and the empirical evidence provided by the recent empirical studies at the industry level among countries by way of applying the theoretical method developed in Turnpike Theory. I will first set up a multi-sector optimal growth model, where each sector exhibits the Harrod neutral technical progress with a sector specific rate. The presented model will be regarded as a multi-sector optimal growth version of the Solow model with the Harrod neutral technical progress.

Secondly, I will rewrite the original model into a per-capita efficiency unit model. Then as the third step, I will transform the efficiency unit model into a reduced form model. Then the method developed in Turnpike Theory are ready to be applicable. I will first establish the Neighborhood Turnpike Theorem demonstrated in McKenzie (1983). The neighborhood Turnpike means that any optimal path will be trapped in a neighborhood of the corresponding optimal steady state path when discount factors are close enough to one and the neighborhood can be made as small as possible by choosing a discount factor arbitrarily close to one. Then, I will show the local stability by applying the logic used by Scheinkman (1976): there exists a stable manifold that stretches out over today's capital stock plane. To demonstrate both theorems, the dynamics of the Neumann-McKenzie facet takes an important role, as we will see later. Combining the Neighborhood Turnpike and the local stability provides the full turnpike property: any optimal path converges to a corresponding optimal steady state when the discount factors are close enough to one. For establishing both theorems, we assume the generalized capital intensity conditions, which are generalized versions of those of a two-sector model. The full turnpike property means that each sector's optimal per-capita capital stock and output converge to its own steady state path with the rate of technical progress determined by the sector's total factor productivity.

The paper is organized in the following manner: In Section 2, I will provide a several empirical facts based on the recent database at the industry level among countries. In Section 3, the model and assumptions are presented and show some existence theorem. In Section 4, the Neumann-McKenzie facet is introduced and the Neighborhood Turnpike Theorem is demonstrated. The results obtained in Section 4 will be used repeatedly in the proofs of main theorems. In Section 5, I show the full Turnpike Theorem. Some comments are given in Section 6.

2 The Empirical Facts

Over the past few years, a great number of efforts have been done to collect and archive the industry level database among countries. Recently such a database is easily accessed on the Web: *the EU-Klems Growth and Productivity Database*[1], which covers 28 countries with 71

(1) URL http://www.euklems.net

industries from 1970 to 2005. It contains the GDPs and the total factor productivity (TFPs) of industries. Growth accounting has been used to analyze economic growth in countries. One of the more interesting applications is to the industries. Let us assume the following production function of the i^{th} industry in a country.

$$Y_i(t) = F^i(K_{1i}(t), K_{2i}(t), \cdots, K_{ni}(t), A_t^i L_i(t)),$$

where Y_i : t^{th} period capital goods output of the i^{th} industry, K_{ji} : i^{th} capital goods used in the j^{th} industry in the t^{th} period, A_t^i : t^{th} period labor-argumented technicalprogress,

and $Li(t)$: t^{th} period labor input of the i^{th} industry. If θ_j stands for the factor share of the j^{th} input factor, then we may derive the following relation concerned with the i^{th} industry;

$$\frac{\dot{A}_i}{A_i} = \frac{\frac{\dot{Y}_i}{Y_i} - \left(\sum_{j=1}^{n} \theta_{ji} \frac{\dot{K}_{ji}}{K_{ji}} + \theta_{0i} \frac{\dot{L}_i}{L_i}\right)}{\theta_{0i}}.$$

Based on this equation, we are able to caluculate TFPs of the 20 industries of a country[(2)]. Figures 1 show the relationship between the per-capita U.S. GDP average growth rate and the U.S. TFP average growth rate at the industry level from 1970 to 2005. In Figures 2, those of the Japanese ecvonomy are exhibited. Note that in both figures, the 45-degree lines are also drawn. If an industry were on the 45-degree line, it would imply that the industry's per-capita GDP would grow at its TFP growth rate. Observing Figures 1 and 2, we may conclude that in both countries, most of the industries lie around the 45-degrees line. Although some industries lie far above or below the 45-degree line.

---

(2) The U.S. 20 industries are followings:
1: TOTAL INDUSTRIES 2: AGRICULTURE, HUNTING, FORESTRY AND FISHING
3: MINING AND QUARRYING 4: TOTAL MANUFACTURING
5: FOOD, BEVERAGES AND TOBACCO 6: TEXTILES, TEXTILE , LEATHER AND FOOTWEAR
7: WOOD AND OF WOOD AND CORK 8: PULP, PAPER, PAPER , PRINTING AND PUBLISHING
9: CHEMICAL, RUBBER, PLASTICS AND FUEL 10: OTHER NON-METALLIC MINERAL
11: BASIC METALS AND FABRICATEDMETAL
12: MACHINERY, NEC 13: ELECTRICAL AND OPTICAL EQUIPMENT
14: TRANSPORT EQUIPMENT 15:MANUFACTURING NEC; RECYCLING
16: ELECTRICITY, GAS AND WATER SUPPLY
17: CONSTRUCTION 18: WHOLESALE AND RETAIL TRADE
19: HOTELS AND RESTAURANTS
20: TRANSPORT AND STORAGE AND COMMUNICATION
For Japanese Economy, three more extra industries are added.

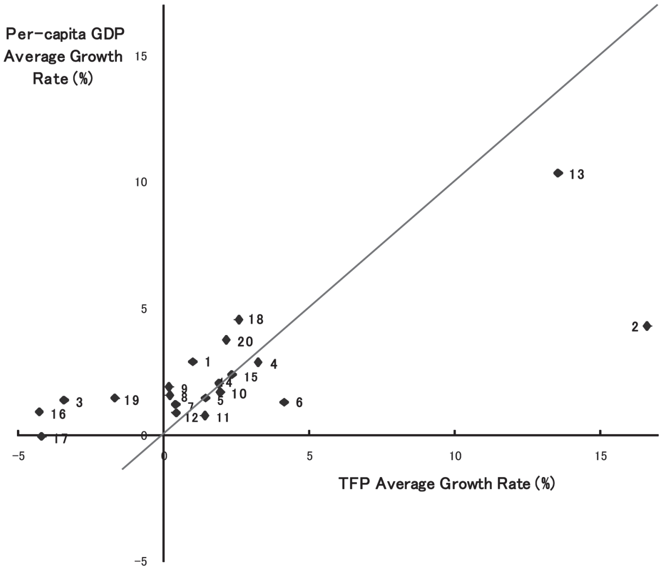

Figure 1: U.S. Economy, 1970-2005 Source: EU-KLEMS DATABASE

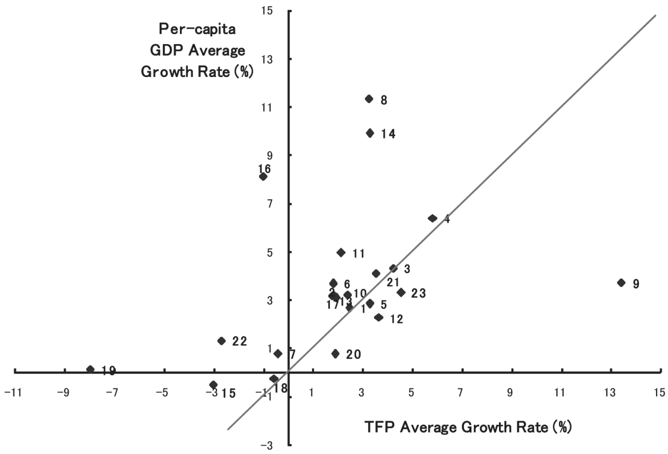

Figure 2: Japanese Economy, 1970-2005 Source: EU-KLEMS DATABASE

We may summarize these facts as follows:

1) Each industrial sector has its own steady state with the sector specific growth rate.

2) The steady state level and its growth rate are highly related to its own TFP.

These facts cannot be explained by the new growth theory totally based on the macro production function. Thus we need to set up an industry based multi-sector growth model. On the other hand, the turnpike theory are established based on the multi-sector model. However it has a drawback, too. The turnpike theory means that each industrial sector with different initial stocks will eventually converge to its own optimal steady state with the common balanced growth rate. In other words, each industry's per capita stock will converges to a certain constant ratio. Thus the turnpike theory cannot explain the facts that each industry's per-capita stock grows at its own growth rate, which is determined by the sectoral TFP.

OECD (2003) also studied the productivity growth at the industry level in detail and reported the following results, which are consistent with our observations discussed above.

- A large contribution to overall productivity growth patterns comes from productivity changes within industries, rather than as a result of significant shifts of employment across industries.
- TFP depends on country/industry specific factors.

From the above discussion, it is an urgent task to set up a multi-sector optimal growth model with technical progress and demonstrate that each sector's per-capita capital and output will grow at the sectoral specific growth rate determined by the sector's TFP.

3 The Model and Assumption

Our model is a discrete-time and multi-sector version of the standard neoclassical optimal growth model with the Harrod neutral technical progress:

$$Max \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t C(t)$$

$$subject\ to:\ k_i(0) = \overline{k}_i$$

$$Y_i(t) + (1-\delta_i)K_i(t) - K_i(t+1) = 0 \tag{1}$$

$$C(t) = F^0(K_{10}(t), K_{20}(t), \cdot\cdot\cdot, K_{n0}(t), A_t^0 L_0(t)), \tag{2}$$

$$Y_i(t) = F^i(K_{1i}(t), K_{2i}(t), \cdot\cdot\cdot, K_{ni}(t), A_t^i L_i(t)), \tag{3}$$

$$\sum_{i=0}^{n} L_i(t) = L(t), \tag{4}$$

$$\sum_{j=0}^{n} K_{ij}(t) = K_i(t), \tag{5}$$

where$i$ = 1, 2, ..., n, t = 0, 1, 2, ..., and the notation is as follows:

r	= subjective rate of discount, r ≥ g,
$C(t) \in R_+$	= total consumption goods produced and consu med at t,
$Y_i(t) \in$ R₊	= tth period capital goods output of the ith sector,

$K_i(t) \in R_+$ = t^{th} period capital stock of the i^{th} sector,

$K_i(0) \in R_+$ = initial capital stock of the i^{th} sector,

$F^j(\cdot): R_+^{n+1} \longrightarrow R_+$ = production function of the j^{th} sector,

$L_i(t)$ = t^{th} period labor input of the i^{th} sector,

$L(t)$ = t^{th} period total labor input,

$K_{ij}(t)$ = i^{th} capital goods used in the j^{th} sector in the t^{th} period,

δ_i = depreciation rate of the i^{th} capital goods, given as $0 < \delta_i < 1$,

A_t^i = t^{th} period labor-argumented technical-progress of the i^{th} sector.

I maintain the following standard assumptions throughout the paper.

Assumption 1. 1) $L(t) = (1+g)^t L(0)$ where g is a rate of population growth and given as $0 < g < 1$. 2) $A_t^i = (1+a_i)^t A_0^i$ where a_i is a rate of labor argumented technical progress of the i th sector and given as $0 < a_i < 1$.

2) of Assumption 1 means that the sectoral TFP is measured by the sectoral labor argumented technical progress (the Harrod neutoral technical progress), which is externally given.

Assumption 2. 1) All the goods are produced nonjointly with production functions F^i ($i=1, \ldots, n$) which are defined on R_+^{n+1}, homogeneous of degree one, strictly quasi-concave and continuously differentiable for positive inputs. 2) Any good j ($j=0, 1, \ldots, n$) cannot be produced unless $K_{ij} > 0$ for some $i = 1, \ldots, n$. 3) Labor must be used directly in each sector. If labor input of some sector is zero, then its sector's output is zero.

Dividing all the variables by A_t^i, we will transform the original model into percapita efficiency unit model. Firstly, let us transform the t^{th} sector's production function as follows; dividing both sides by $A_t^i L(t)$,

$$\frac{Y_i(t)}{A_t^i L(t)} = F^i\left(\frac{K_{1i}(t)}{A_t^i L(t)}, \frac{K_{2i}(t)}{A_t^i L(t)}, \cdots, \frac{K_{ni}(t)}{A_t^i L(t)}, \frac{A_t^i L_i(t)}{A_t^i L(t)}\right) \ (i = 1, \cdots, n).$$

Then,

$$\widetilde{y}_i(t) = f^i(\widetilde{k}_{1i}(t), \widetilde{k}_{2i}(t), \cdots, \widetilde{k}_{ni}(t), \ell_i(t)) \ \ (i = 1, \cdots, n)$$

where $\widetilde{y}_i(t) = \frac{Y_i(t)}{A_t^i L(t)}, \widetilde{k}_{1i}(t) = \frac{K_{1i}(t)}{A_t^i L(t)}, \widetilde{k}_{2i}(t) = \frac{K_{2i}(t)}{A_t^i L(t)}, \cdots, \widetilde{k}_{ni}(t) = \frac{K_{ni}(t)}{A_t^i L(t)}$,and $\ell_i(t) = \frac{A_t^i L_i(t)}{A_t^i L(t)}$.

Applying the same transformation to the consumption sector, we have also

$$\widetilde{c}(t) = f^0(\widetilde{k}_{10}(t), \widetilde{k}_{20}(t), \cdots, \widetilde{k}_{n0}(t), \ell_0(t)).$$

Furthermore, we may also transform the t^{th} sector's accumulation equation as followas; dividing both sides by $A_t^i L(t)$,

$$\frac{Y_i(t)}{A_t^i L(t)} + (1-\delta_i)\frac{K_i(t)}{A_t^i L(t)} - \frac{K_i(t+1)}{A_t^i L(t)} = 0$$

Note the following relation:

$$\frac{K_i(t+1)}{A_t^i L(t)} = \frac{(1+a_i)(1+g)K_i(t+1)}{[(1+a_i)A_t^i][(1+g)L(t)]} = (1+a_i)(1+g)\widetilde{k}_i(t+1).$$

Then we have finally,

$$\widetilde{y}_i(t) + (1-\delta_i)\widetilde{k}_i(t) - (1+a_i)(1+g)\widetilde{k}_i(t+1) = 0.$$

In a vector form expression,

$$\widetilde{\mathbf{y}} + (\mathbf{I}-\Delta)\widetilde{\mathbf{k}}(t) - (1+g)\mathbf{G}\widetilde{\mathbf{k}}(t+1) = 0$$

where G and Δ are following diagonal matices:

$$\mathbf{G} = \begin{pmatrix} (1+a_i) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (1+a_n) \end{pmatrix} \text{ and } \Delta = \begin{pmatrix} \delta_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \delta_n \end{pmatrix}.$$

We can also rewrite the objective function in terms of per-capita as follows;

$$\widetilde{c}(t) = \frac{C(t)}{A_t^0 L(t)} = \frac{C(t)}{(1+a_i)^t(1+g)^t A_0^0 L(0)}.$$

Then,

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t C(t) = \sum_{t=0}^{\infty}\left[\frac{(1+g)(1+a_0)}{(1+r)}\right]^t \widetilde{c}(t)$$

Now the original model can be rewritten as the following per-capita efficiency unit model:

The Per-capita Efficiency Unit Model:

$$Max\ \rho^t \widetilde{c}(t) \ \ where\ \rho = \frac{(1+g)(1+a_0)}{(1+r)},$$

$$s.t.\ \ \widetilde{k}_i(0) = \overline{k}_i \ \ (i = 1, \cdots, n),$$

$$\widetilde{c}(t) = f^0(\widetilde{k}_{10}(t), \widetilde{k}_{20}(t), \cdots, \widetilde{k}_{n0}(t), \ell_0(t)), \tag{6}$$

$$\widetilde{y}_i(t) = f^i(\widetilde{k}_{1i}(t), \widetilde{k}_{2i}(t), \cdots, \widetilde{k}_{ni}(t), \ell_i(t)) \ (i = 1, \cdots, n), \tag{7}$$

$$\widetilde{\mathbf{y}} + (\mathbf{I} - \Delta)\widetilde{\mathbf{k}}(t) - (1+g)\mathbf{G}\widetilde{\mathbf{k}}(t+1) = 0, \tag{8}$$

$$\sum_{i=0}^{n} \ell_i(t) = 1, \tag{9}$$

$$\sum_{i=0}^{n} \widetilde{k}_{ij}(t) = \widetilde{k}_j(t) \ (j = 1, \cdots, n). \tag{10}$$

We may add the following assumption and prove the basic property, Lemma 1;

Assumptin 3. $0 < \rho < 1$.

Lemma 1. Under Assumption 2, Eqs.(6)-(10) except Eq.(8) are summarized as the social production function $\widetilde{c}(t) = T(\widetilde{\mathbf{y}}(t), \widetilde{\mathbf{k}}(t))$ which is continuously differentiable on the interior $\mathbf{R}_+^{2n}$ and concave where $\widetilde{\mathbf{y}}(t) = (y_1(t), y_3(t), ..., y_n(t))$ and $\widetilde{\mathbf{k}}(t) = (k_1(t), k_2(t), ..., k_n(t))$.

Proof.

See Benhabib and Nishimura (1979).

If $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{z}}$ stand for initial and terminal capital stock vectors respectively, then the reduced form utility function V $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{z}})$ and the feasible set D can be defined as follows:

$$V(\widetilde{\mathbf{x}}, \widetilde{\mathbf{z}}) = T[(1+g)\mathbf{G}\widetilde{\mathbf{z}} - (\mathbf{I} - \Delta)\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}]$$

and

$$D = \{(\widetilde{\mathbf{x}}, \widetilde{\mathbf{z}}) \in R_+^n \times R_+^n : T[(1+g)\mathbf{G}\widetilde{\mathbf{z}} - (\mathbf{I} - \Delta)\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}] \geq 0\}$$

where $\widetilde{\mathbf{x}} = (\widetilde{x}_1(t), \widetilde{x}_2(t), ..., \widetilde{x}_n(t))$, $\widetilde{\mathbf{z}} = (\widetilde{\mathrm{k}}_1(t+1), \widetilde{k}_2(t+1), ..., \widetilde{\mathrm{k}}_n(t+1))$ and $\mathbf{I}$ is an n-dimensional unit matrix.

Finally, the above optimization problem will be summarized as the following standard reduced form problem, which is familiar in the Turnpike Theory:

Reduced Form Model

$$Maximize\ \sum_{t=0}^{\infty} \rho^t V(\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t+1))$$

$$subject\ to\ (\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t+1)) \in D\ \ for\ t \geq 0\ and\ \widetilde{\mathbf{k}}(0) = \overline{\mathbf{k}}.$$

Also note that any interior optimal path must satisfy the following *Euler Equations*, showing an intertemporal efficiency allocation:

$$\mathbf{V}_z(\widetilde{\mathbf{k}}(t-1), \widetilde{\mathbf{k}}(t)) + \rho \mathbf{V}_x(\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t+1)) = \mathbf{0} \text{ for all } t \geq 0 \tag{11}$$

where the partial derivative vectors mean that

$$\begin{aligned} \mathbf{V}_x(\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t+1)) &= [\partial V(\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t+1))/\partial \widetilde{k}_1(t), \cdots, \partial V(\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t+1))/\partial \widetilde{k}_n(t)]^t, \\ \mathbf{V}_z(\widetilde{\mathbf{k}}(t-1), \widetilde{\mathbf{k}}(t)) &= [\partial V(\widetilde{\mathbf{k}}(t-1), \widetilde{\mathbf{k}}(t))/\partial \widetilde{k}_1(t), \cdots, \partial V(\widetilde{\mathbf{k}}(t), \widetilde{\mathbf{k}}(t-1))/\partial \widetilde{k}_n(t)]^t \end{aligned}$$

and $\mathbf{0}$ means an n dimensional zero column vector. " t " implies transposition of vectors. Note that under the differentiability assumptions, all the price vectors will satisfy the following relations:

$$\begin{aligned} q &= \partial \widetilde{c}/\partial \widetilde{c} = 1, \\ p_i &= -q \partial T(\widetilde{\mathbf{y}}, \widetilde{\mathbf{k}})/\partial \widetilde{k}_i \quad (i = 1, 2, \cdots, n), \\ w_i &= q \partial T(\widetilde{\mathbf{y}}, \widetilde{\mathbf{k}})/\partial \widetilde{k}_i \quad (i = 1, 2, \cdots, n). \\ w_0 &= q\widetilde{c} + \mathbf{p}\widetilde{\mathbf{y}} - \mathbf{w}\widetilde{\mathbf{k}} \end{aligned}$$

Using these relation, we may define the price vectors of capital goods as $(n \times 1)$ row vector $\mathbf{p} = (p_1, p_2, ..., p_n)$, the output of capital goods as $(n \times 1)$ vector $\widetilde{\mathbf{y}} = (\widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_n)^t$, the rental rate as $(1 \times n)$ row vector $\mathbf{w} = (w_1, w_2, ..., w_n)$ and the capital stock as $(n \times 1)$ vector $\widetilde{\mathbf{k}} = (\widetilde{k}_1, \widetilde{k}_2, ..., \widetilde{k}_n)^t$. w_0 is a wage rate. For simlicity we may assume that all the price vectors $(\mathbf{p}, \mathbf{w}, w_0)$ are expressed as the relative price vectors of the price of the consumption good q.

Definition. *An optimal steady state path* $\mathbf{k}^\rho$ (denoted by OSS henceforth) is an optimal path which solves the above optimization problem and $\widetilde{\mathbf{k}}^\rho = \widetilde{\mathbf{k}}(t) = \widetilde{\mathbf{k}}(t+1)$ for all $t \geq 0$.

Due to the homogenety assumption of each sector's production, it is often convenient to express a chosen technology as a technology matrix. Let us define the technology matrix as follows:

$$\mathbf{A} = \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ a_{10} & & \\ \vdots & \overline{\mathbf{A}} & \\ a_{n0} & & \end{pmatrix} = \begin{pmatrix} a_{00} & \mathbf{a}_{0.} \\ \mathbf{a}_{.0} & \overline{\mathbf{A}} \end{pmatrix}$$

where $a_{0i} = \widetilde{\ell}_i/\widetilde{y}_i \ (i = 0, \cdots, n)$, $a_{ij} = \widetilde{k}_{ij}/\widetilde{y}_j \ (i = 1, \cdots, n; j = 0, 1, \cdots, n)$ and

$$\overline{\mathbf{A}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

It directly follows that Assumption 2 implies that for all $j = 0, 1, \ldots, n$, $a_{ij} > 0$ for some $i = 1, \ldots, n$ and $a_{0i} > 0$ for all i. We make first the following assumption in terms of the technology matrix to show the existence theorem.

Assumption 4. (*Viability*) For a given r ($\geq g$), a chosen technology matrix $\overline{\mathbf{A}}$ satisfies

$$[\mathbf{I} - (r\mathbf{I} + \Delta)\overline{\mathbf{A}}^r]^{-1} \geq \Theta$$

where Θ is a $n \times n$ zero matrix[3].

By the well known equivalence theorem of the Hawkins-Simon condition and Theorem 4 of Mckenzie (1960), Assumption 4 is equivalent to the property that the matrix $[\mathbf{I} - (r\mathbf{I} + \Delta)\overline{\mathbf{A}}^r]$ has a dominant diagonal that is positive; there exists $\mathbf{y} \geq \mathbf{0}$ such that $[\mathbf{I} - (r\mathbf{I} + \Delta)\overline{\mathbf{A}}^r] \geq \mathbf{0}$.

We need the following extra assumption.

Assumption 5. $1 > a_0 > \max_{i=1,\ldots,n} |a_i|$

Remark 1 *The assumption means that the TFP growth rate of the consumption sector is the highest among those of sectors. Takahashi, Mashiyama and Sakagami (2004) reported that in the postwar Japanese economy, the consumption sector has exhibited a higer per-capita output growth rate than that of the capital goods sector in a two-sector model. If the TFP growth rate has a positive correlation with the per-capita sectoral GDP growth rate, this fact will partially support Assumption 5.*

(3) Let $\mathbf{A}$ and Θ be n-dimentional square matrix and n-dimensional zero matrix. Then $\mathrm{A} \gg \Theta$ if $a_{ij} > 0$ for all i, j, $\mathbf{A} > \Theta$ if $a_{ij} \geq 0$ for all i, j and $a_{ij} > 0$ for some i, j and $\mathbf{A} \geq \Theta$ if $a_{ij} \geq 0$ for all i, j.

McKenzie (1983, 1984) has shown that the existence of an optimal path and OSS in the reduced form model. Actually we can prove the following existence theorem under Assumptions 1 through 5.

Existence Theorem: Under Assumptions 1 through 5, there exists an optimal steady state path $\tilde{\mathbf{k}}^{\rho}$ for $\rho \in (0, 1]$ and an optimal path $\{\tilde{\mathbf{k}}^{\rho}(t)\}^{\infty}$ from any sufficient initial stock vector $\tilde{\mathbf{k}}(0)$[(4)].

Proof. We need to show that under Assumptions 1 through 3, all the conditions[(5)] in Theorem 1 of McKenzie (1983) or in the existence theorem of McKenzie (1984) are satisfied. Especially, Assumption 4 and the additional condition are needed to guarantee the non-emptiness of the interior of D (Condition 5) in the footnote) as we will demonstrate as follows; from the Condition 5), there is an output vector $\mathbf{y} \geq \mathbf{0}$ such that $[\mathbf{I} - (r\mathbf{I} + \Delta)\overline{\mathbf{A}}^{r}] \geq \mathbf{0}$. By a scalar multiplication of $\mathbf{y}$, we can establish $\hat{\mathbf{x}} = \mathbf{A}^{r}\hat{\mathbf{y}}$ where $\hat{\mathbf{x}} = (1, \overline{\mathbf{x}})^{t}$ and $\hat{\mathbf{y}} = (c, \overline{\mathbf{y}})$. Note that the equality of the first elements of $\hat{\mathbf{x}}$ and $\mathbf{A}^{r}\hat{\mathbf{y}}$ will provide Eq. (9); the full employment condition. Since the labor constraints are satisfied for $\hat{\mathbf{y}}$ and that $\overline{\mathbf{A}}^{r}$ is a submatrix of $\mathbf{A}^{r}$, it follows that $\overline{\mathbf{x}} = \overline{\mathbf{A}}^{r}\hat{\mathbf{y}}$ holds.

$$
\begin{aligned}
\overline{\mathbf{z}} - \rho^{-1}\overline{\mathbf{x}} &= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}\left\{I + [\mathbf{I} - \overline{\Delta} - (1+g)\mathbf{G}\rho^{-1}]\overline{\mathbf{A}}^{r}\right\}\overline{\mathbf{y}} \\
&= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}I + \mathbf{I} - \Delta - (1+g)\begin{pmatrix} (1+a_1) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (1+a_n) \end{pmatrix} \\
&\qquad \left[\frac{(1+r)}{(1+g)(1+a_0)}\right]\Bigg]\overline{\mathbf{A}}^{r}\Bigg\}\overline{\mathbf{y}} \\
&= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}\left\{I + \left[\mathbf{I} - \Delta - (1+r)\begin{pmatrix} \frac{(1+a_1)}{(1+a_0)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \frac{(1+a_n)}{(1+a_0)} \end{pmatrix}\right]\overline{\mathbf{A}}^{r}\right\}\overline{\mathbf{y}} \\
&\geq \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}\left\{I + [\mathbf{I} - \Delta - (1+r)\mathbf{I}]\overline{\mathbf{A}}^{r}\right\}\overline{\mathbf{y}} \quad \text{due to Assumption 5,}
\end{aligned}
$$

(4) A capital stock x is called sufficient if there is a finite sequence (k(0), k(1), ..., k(T)) where x=k(0), (k(t), k(t+1)) $\in D$ and k(T) is expansible. k(T) is expansible if there is k(T+1) such that k(T+1)$\gg$k(T) and (k(T), k(T+1))$\in D$. Note that the sufficiency will be assured by assuming "Inada-type" condition on the production functions.

(5) McKenzie's conditions are followings: 1) $V(\mathbf{x}, \mathbf{z})$ are defined on a convex set D. 2) There is a $\eta > 0$ such that $(\mathbf{x}, \mathbf{z}) \in D$ and $|\mathbf{z}| < \xi < \infty$ implies $|\mathbf{z}| < \eta < \infty$. 3) If $(\mathbf{x}, \mathbf{z}) \in D$, then $(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) \in D$ for all $\tilde{\mathbf{x}} \geq \mathbf{x}$ and $0 \leq \tilde{\mathbf{z}} \leq \mathbf{z}$. Moreover $V(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) \geq V(\mathbf{x}, \mathbf{z})$. 4) Ther is $\zeta > 0$ such that $|\mathbf{x}| \geq \zeta$ implies for any $(\mathbf{x}, \mathbf{z}) \in D$, $|\mathbf{z}| < \lambda\ |\mathbf{x}|$ where $0 < \lambda < 1$. 5) There is $(\overline{\mathbf{x}}, \overline{\mathbf{z}}) \in D$ such that $\rho\overline{\mathbf{z}} > \overline{\mathbf{x}}$.

$$= \left(\frac{1}{1+g}\right)\mathbf{G}^{-1}[\mathbf{I}-(r\mathbf{I}+\overline{\Delta})]\overline{\mathbf{A}}^{r}\overline{\mathbf{y}} > 0 \text{ from Assumption 4,}$$

Therefore $\overline{\mathbf{y}}$ will be chosen so that $\overline{\mathbf{z}}-\rho^{-1}\overline{\mathbf{x}}\geq\mathbf{0}$ where $(\overline{\mathbf{x}}, \overline{\mathbf{z}})\ \varepsilon D$. See also Lemma 3 through Lemma 7 in Takahashi (1985).

Remark 2 *It should be noted that since* $\tilde{k}_i^{\rho}=\frac{k_i^{\rho}(t)}{A_t^i L_t}$, *it follows that* $k_i^{\rho}(t)=\tilde{k}_i^{\rho}A_t^i=(1+a_i)^t A_0^i\tilde{k}_i^{\rho}$ *for* $i=1, ..., n$. *Thus the original series of the industry's optimal per-capita stock* $\tilde{k}_i^{\rho}(t)$ *is growing at the rate of its own sector's technical progress,* $(1+a_i)$. *From now on, to avoid further complications of our nortation, all the variables measured in efficiency unit will be denoted without the simbol "~" unless otherwise mentioned.*

Suppose that $\mathbf{k}^{\rho}$ is an interior OSS in efficiency uite with a given ρ, then it must satisfy the Euler equations:

$$\mathbf{V}_z(\mathbf{k}^{\rho},\mathbf{k}^{\rho})+\rho\mathbf{V}_x(\mathbf{k}^{\rho},\mathbf{k}^{\rho})=\mathbf{0}. \tag{12}$$

Due to the above definition of OSS, we will express the partial derivatives of the Euler equations in terms of price vectors:

$$\begin{aligned}\mathbf{V}_x(\mathbf{k}^{\rho},\mathbf{k}^{\rho}) &= \mathbf{p}^{\rho}(\mathbf{I}-\Delta)+\mathbf{w}^{\rho}\\ \mathbf{V}_z(\mathbf{k}^{\rho},\mathbf{k}^{\rho}) &= -(1+g)\mathbf{G}\mathbf{p}^{\rho}.\end{aligned}$$

where $\mathbf{I}$ is a $n\times n$ unit matrix. Substituting these relations into the Euler equations may yield the followings:

$$\rho[\mathbf{w}^{\rho}+\mathbf{p}^{\rho}(\mathbf{I}-\Delta)]-(1+g)\mathbf{G}\mathbf{p}^{\rho}=\mathbf{0} \tag{13}$$

and further calculation will finally yield:

$$\mathbf{p}^{\rho}\left[-\mathbf{I}+\Delta+\left(\frac{1+r}{1+a_0}\right)\mathbf{G}\right]=\mathbf{w}^{\rho}.$$

These are clearly non-arbitrage conditions among capital goods and imply that any capital good must yield the same rate of returns as the subjective discount rate ρ. Thus the Euler conditions are the non-arbitrage conditions.

Because of the differentiability and the constant returns to scale technologies, the well-known proposition proved by Samuelson (1945) will hold: the cost function denoted by $C^i(\mathrm{w}_0, \mathbf{w}^{\rho})$ $(i=1, ..., n)$ is homogeneous of degree one and $\partial C^i/\partial w_j=a_{ij}$ where $a_{ij}=k_{ij}/y_j$ $(i=1, 2, ..., n;$ $j=0, 1, ..., n)$. Due to the cost minimization condition and this property, a unique technology matrix $\mathbf{A}^{\rho}$ is chosen along the OSS $\mathbf{k}^{\rho}$. Also note that due to Assumption 3, for a given $\rho\in(0, 1]$, the uniquely chosen technology matrix $\overline{\mathbf{A}}^{\rho}$ along the OSS $\mathbf{k}^{\rho}$ have to satisfy,

$$[\mathbf{I}-(r\mathbf{I}+\overline{\Delta})\overline{\mathbf{A}}^{\rho}]^{-1}\geq\Theta.$$

Furthermore it follows that $a^{\rho}_{00}>0$ and $\mathbf{a}^{\rho}_{\mathbf{0}\cdot}\gg\mathbf{0}$ from Assumption 2. Henceforth, we use the symbol " ρ " to clarify that vectors and matrices are evaluated along OSS $\mathbf{k}^{\rho}$. Conbining these results, the following important property will be established:

Lemma 2. When $\rho\in(0,1]$, there exists a unique OSS $(\mathbf{k}^{\rho}\gg 0)$(6) with the corresponding unique positive price vector $\mathbf{p}^{\rho}$ and positive factor price vector $(\mathrm{w}^{\rho}_{0},\mathbf{w}^{\rho})$.

Proof. We can apply the same argument as the one used in Theorem1 of Burmeister and Grahm (1975).

From this lemma, along OSS with ρ, the nonsingular technology matrix $\mathbf{A}^{\rho}$ is chosen and the cost-minimization condition and the full-employment condition will be expressed as follows:

$$(1,\mathbf{p}^{\rho})=(w^{\rho}_{0},\mathbf{w}^{\rho})\mathbf{A}^{\rho},$$

and

$$(1,\mathbf{k}^{\rho})^{t}=\mathbf{A}^{\rho}(c^{\rho},\mathbf{y}^{\rho})^{t}.$$

If $\mathbf{A}^{\rho}$ has the inverse matrix $\mathbf{B}^{\rho}$, then solving the above conditions respectively yields,

$$\mathbf{p}^{\rho}=\mathbf{w}^{\rho}\left(\mathbf{a}^{\rho}-\frac{1}{a^{\rho}_{00}}\mathbf{a}^{\rho}_{\cdot\mathbf{0}}\mathbf{a}^{\rho}_{\mathbf{0}\cdot}\right)+\frac{\mathbf{a}^{\rho}_{\mathbf{0}\cdot}}{a^{\rho}_{00}}=\mathbf{w}^{\rho}(\mathbf{b}^{\rho})^{-1}+\frac{\mathbf{a}^{\rho}_{\mathbf{0}\cdot}}{a^{\rho}_{00}}$$

and

$$(\mathbf{k}^{\rho})^{t}=\left(\mathbf{a}^{\rho}-\frac{1}{a^{\rho}_{00}}\mathbf{a}^{\rho}_{\cdot\mathbf{0}}\mathbf{a}^{\rho}_{\mathbf{0}\cdot}\right)(\mathbf{y}^{\rho})^{t}+\frac{\mathbf{a}^{r}_{\cdot\mathbf{0}}}{a^{\rho}_{00}}=(\mathbf{b}^{\rho})^{-1}(\mathbf{y}^{\rho})^{t}+\frac{\mathbf{a}^{r}_{\cdot\mathbf{0}}}{a^{\rho}_{00}}$$

where $\mathbf{b}^{\rho}$ is a submatrix of $\mathbf{B}^{\rho}$ defined as follows:

$$\mathbf{B}^{\rho}=(\mathbf{A}^{\rho})^{-1}=\begin{pmatrix}b^{\rho}_{00} & \mathbf{b}^{\rho}_{0\cdot}\\ \mathbf{b}^{\rho}_{\cdot 0} & \mathbf{b}^{\rho}\end{pmatrix}.$$

And the nonsingularity of $\mathbf{b}^{\rho}$ comes from the following observation: From Murata (1977), $\mathbf{b}^{\rho}=[\mathbf{a}^{\rho}-(1/a^{\rho}_{00})\mathbf{a}^{\rho}_{\cdot 0}\mathbf{a}^{\rho}_{0\cdot}]^{-1}$. Furthermore, by Gantmacher (1960), it also follows that det $\mathbf{A}^{\rho}=a^{\rho}_{00}$det $[\mathbf{a}^{\rho}_{\cdot 0}\mathbf{a}^{\rho}_{0\cdot}]$. Since $\mathbf{A}^{\rho}$ is non-singular, the result follows.

(6) Let $\mathbf{x}$ and $\mathbf{y}$ be n-dimensional vectors. Then $\mathbf{x}\gg\mathbf{y}$ if $x_i>y_i$ for all i, $\mathbf{x}>\mathbf{y}$ if $x_i\geq y_i$ for all i and at least one j, $x_i>y_i$ and $\mathbf{x}\geq\mathbf{y}$ if $x_i\geq y_i$ for all i.

From now on, we are concentrated on the OSS with $\rho = 1$ denoted by $\mathbf{k}^*$. We will also use "$*$" to denote the elements and variables are evaluated at $\mathbf{k}^*$.

Definition. When $\rho = 1$, the chosen technology matrix $\mathbf{A}^*$ satisfies the *Generalized Capital Intensity GCI-I* condition, if there exists a set of positive number $(d_1, \ldots, d_n)$ such that

$$d_s(\frac{a^*_{ss}}{a^*_{0s}} - \frac{a^*_{s0}}{a^*_{00}}) > \sum_{i \neq s,0}^{n} d_i \left| \frac{a^*_{si}}{a^*_{0i}} - \frac{a^*_{s0}}{a^*_{00}} \right| \quad for\ s = 1, \cdots, n.$$

Similarly, the technology matrix $\mathbf{A}^*$ satisfies the *Generalized Capital Intensity GCI-II* condition, if there exists a set of positive number $(d_1, \ldots, d_n)$ such that

$$\frac{a^*_{ss}}{a^*_{0s}} - \frac{a^*_{s0}}{a^*_{00}} < 0$$

and

$$d_s \left| \frac{a^*_{ss}}{a^*_{0s}} - \frac{a^*_{s0}}{a^*_{00}} \right| > \sum_{i \neq s,0}^{n} d_i \left| \frac{a^*_{si}}{a^*_{0i}} - \frac{a^*_{s0}}{a^*_{00}} \right| \quad for\ s = 1, \cdots, n.$$

Consider a capital good sector s, and focus on its own capital input s and its capital-labor ratio in all the other sectors. By the definition the left-hand side of the GCI-I condition means the excess of the capital-labor ratio of capital input s for the capital good sector s. The right-hand side collects the absolute values of the discrepancy between the capital-labor ratio of other sectors $(i \neq s, 0)$ to that of the consumption sector. The GCI-I condition points that the sum of such absolute values still fall short of the the excess of a^*_{ss}/a^*_{0s} over the comparable ratio in the consumption sector, a^*_{ss}/a^*_{0s}. We may give a similar explanation to the GCI-II condition.

The following lemma can be directly derived from the definition of the both intensity conditions.

Lemma 3. If the technology matrix $\mathbf{A}^*$ satisfies the GCI-I (GCI- II condition) condition, its inverse matrix $\mathbf{B}^*$ has positive (negative) diagonal ellements and negative (positive) off-diagonal elements.

Proof. From the theorem by Jones et al. (1993), under the Strong GCI-II, its inverse matrix has negative diagonal and positive off-diagonal elements. On the other hand, under the Strong CGI-I, by considering the case where one price falls with all the other prices constant in their proof, their exact logic can be applicable and the first result will be derived.

Due to Lemma 3, we may prove the following important lemma.

Lemma 4. Under the Strong GCI-I (the Strong GCI-II), $[\mathbf{b}^* - ((1+g)\mathbf{G} + \Delta - \mathbf{I})]$ has a dominant diagonal that is positive (negative) for rows[(7)].

Proof. Due to Lemma 3, under the Strong GCI-I (the Strong GCI-II), the inverse matrix $\mathbf{B}^*$ has positive (negative) diagonal elements and negative (positive) offdiagonal elements. From the accumulation equation $\mathbf{y}^* = (1+n)\mathbf{G}\mathbf{k}^* - (\mathbf{I} - \Delta)\mathbf{k}^*$ and $\mathbf{y}^* = \mathbf{b}^*\mathbf{k}^* + \mathbf{b}^*$. it follows that

$$[\mathbf{b}^* - ((1+g)\mathbf{G} + \Delta - \mathbf{I})]\mathbf{k}^* = -\mathbf{b}^*_{.0}$$

Due to Lemma 3, $-\mathbf{b}^*_{.0} < (>) \mathbf{0}$. Theirfore the matrix $[\mathbf{b}^* - ((1+g)\mathbf{G} + \Delta - \mathbf{I})]$ has the negative (positive) dominant diagonal for rows.

From now on we may call the dominant diagonal that is negative as the n.d.d. and also call the dominant diagonal that is positive as the p.d.d. for short.

From the Euler equations (12), its Jacobian $\mathbf{J}(\mathbf{k}, \rho)$ is

$$\mathbf{J}(\mathbf{k}, \rho) = \rho \mathbf{V}_{xx}(\mathbf{k}, \mathbf{k}) + \rho \mathbf{V}_{xz}(\mathbf{k}, \mathbf{k}) + \mathbf{V}_{zx}(\mathbf{k}, \mathbf{k}) + \mathbf{V}_{zz}(\mathbf{k}, \mathbf{k}),$$

which at $\mathbf{k}^*$ is

$$\mathbf{J}(\mathbf{k}, 1) = \mathbf{V}_{zz}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{xz}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{zx}(\mathbf{k}^*, \mathbf{k}^*) + \mathbf{V}_{xx}(\mathbf{k}^*, \mathbf{k}^*)$$

where all matrices are evaluated at $\mathbf{k}^*$[(8)]. We will show the following important lemma, which is corresponding to Lemma 2.5 of Takahashi (1992).

Lemma 5. Suppose that either of the GCI conditions hold. Then there exists a positive scalar $\bar{\rho}$ such that for $\rho \in [\bar{\rho}, 1]$, the OSS $\mathbf{k}^\rho$ is unique and is a continuous vector-value function of ρ, namely $\mathbf{k}^\rho = \mathbf{k}(\rho)$.

Proof. If $\det \mathbf{J}(\mathbf{k}^*, 1) \neq 0$ holds, then due to the Implicit Function Theorem, the result follows. To show this we will use the following fact shown in Benhabib and Nishimura (1979):

$$\mathbf{T}_1 = [\partial T/\partial \mathbf{y}] = -\mathbf{p}, \ \mathbf{T}_2 = [\partial T/\partial \mathbf{k}] = \mathbf{w}$$

where $\mathbf{p}$ is an output price vector. Differentiating both price vectors with resoect to $\mathbf{y}$ and $\mathbf{k}$ again will yield the following second-order partial derivative matrices: $\mathbf{T}_{11} = [-\partial \mathbf{p}/\partial \mathbf{y}]$, $\mathbf{T}_{12} = [-\partial \mathbf{p}/\partial \mathbf{k}]$, $\mathrm{T}_{21} = [\partial \mathbf{w}/\partial \mathbf{y}]$ and $\mathrm{T}_{22} = [\partial \mathbf{w}/\partial \mathbf{k}]$. Also note that if the matrices are evaluated at $\mathbf{k}^*$, then from the previous equation,

$$[\partial \mathbf{p}/\partial \mathbf{w}] = (\mathbf{b}^*)^{-1}$$

(7) Suppose $\mathbf{A}$ is an $n \times n$ matrix and its diagonal elements are negative (positive). Let there exist a positive vector $\mathbf{h}$ such that $h_i|a_{ii}| > \Sigma^n_{j=1, j\neq i} h_j |a_{ij}|$, $i = 1, 2, ..., n$. Then $\mathbf{A}$ is said to have a dominant main diagonal that is negative (positive) for rows. See McKenzie (1960) and Murata (1977).

(8) We use the following notational convention for the partial derivative matrices: $\mathrm{V}_{xx} = [\partial^2 \mathbf{V}(\mathbf{x}, \mathbf{z})/\partial \mathrm{x}^2]$, $\mathrm{V}_{xz} = [\partial^2 \mathbf{V}(\mathbf{x}, \mathbf{z})/\partial \mathbf{x} \partial \mathbf{z}]$ and $\mathrm{V}_{zz} = [\partial^2 \mathbf{V}(\mathbf{x}, \mathbf{z})/\partial \mathbf{z}^2]$. Note that each matrix is an $n \times n$ matrix.

and due to the symmetry of the Hessian matrix of $c(t)=T(\mathbf{y}(t), \mathbf{k}(t))$,

$$[\partial \mathbf{p}/\partial \mathbf{k}] = -[\partial \mathbf{w}/\partial \mathbf{y}]^t$$

where the suffix "$'$" means a transpose of a matrix. Utilizing these relations, all the partial derivative matrices at $\mathbf{k}^*$ can be expressed in terms of the matrices $\mathbf{b}^*$ and $\mathbf{T}_{22}$ as follows: $\mathbf{T}_{11}=(\mathbf{b}^*)^{-1}\mathbf{T}_{22}^t(\mathbf{b}^*)^{-1}=(\mathbf{b}^*)^{-1}\mathbf{T}_{22}(\mathbf{b}^*)^{-1}$, $\mathbf{T}_{12}=-(\mathbf{b}^*)^{-1}\mathbf{T}_{22}$, and $\mathbf{T}_{21}=-\mathbf{T}_{22}(\mathbf{b}^*)^{-1}$. Substituting $\mathbf{Y}_x=(g\mathbf{I}+\Delta)$ and $\mathbf{Y}_z=\mathbf{I}$ into Eq.(2.22) of Takahashi (1985), the Jacobian will be expressed as follows:

$$\mathbf{J}(\mathbf{k}^*, 1) = [(1+g)\mathbf{G}+\Delta-\mathbf{I}, \mathbf{I}] \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} (1+g)\mathbf{G}+\Delta-\mathbf{I} \\ \mathbf{I} \end{pmatrix}.$$

If the righthand side is negative definite, then the proof will be completed. Substituting all the relations obtained before into the Hessian matrix of the social production function and suppose that the matrix $\mathbf{b}^*$ is nonsingular, then we may yield the following equation:

$$\begin{aligned} & [(1+g)\mathbf{G}+\Delta-\mathbf{I}, \mathbf{I}] \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} (1+g)\mathbf{G}+\Delta-\mathbf{I} \\ \mathbf{I} \end{pmatrix} \\ = \; & ((1+g)\mathbf{G}+\Delta-\mathbf{I})\mathbf{T}_{11}((1+g)\mathbf{G}+\Delta-\mathbf{I}) \\ & +((1+g)\mathbf{G}+\Delta-\mathbf{I})\mathbf{T}_{21}+\mathbf{T}_{12}((1+g)\mathbf{G}+\Delta-\mathbf{I})+\mathbf{T}_{22} \\ = \; & [\mathbf{b}^*-((1+g)\mathbf{G}+\Delta-\mathbf{I})]^2[(\mathbf{b}^*)^{-1}]^2\mathbf{T}_{22}. \end{aligned}$$

Due to Lemma 4, the matrix $[\mathbf{b}^*((1+g)\mathbf{G}+\Delta-\mathbf{I})]$ has the negative (positive) d.d. from the GCI conditions and it must be nonsingular. $\mathbf{b}^*$ is also nonsingular. $\mathbf{T}_{22}$ is negative definite and nonsingular due to the argument of Benhabib and Nishimura (1979, pp68-69). Furthermore, the first two matrices are symmetric and therefore all the elements are positive.Thus the above matrix is negative definite and the proof is completed.

From this lemma, it follows that all the price vectors $\mathbf{p}^\rho$, $\mathbf{w}^\rho$, and the technology matrix $\mathbf{A}^\rho$ are continuous vecto-value functions of $\rho \in [\rho', 1]$.

4 The Neumann-McKenzie Facet

Now we will introduce the Neumann-McKenzie Facet (NMF for short), which plays an important role in stability arguments regarding neoclassical growth models as studied in

Takahashi (1985) and Takahashi (1992) and has been intensively studied by L. McKenzie (see especially McKenzie (1983)). The NMF will be defined in the reduced form model as follows:

Definition. *The Neumann-McKenzie Facet* of an OSS, denoted by $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$, is defined as:

$$\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho) = \{(\mathbf{x}, \mathbf{z}) \in D : c + \rho\mathbf{p}^\rho\mathbf{z} - \mathbf{p}^\rho\mathbf{x} = c^\rho + \rho\mathbf{p}^\rho\mathbf{k}^\rho - \mathbf{p}^\rho\mathbf{k}^\rho\},$$

where $\mathbf{k}^\rho$ is an OSS and $\mathbf{p}^\rho$ is a supporting price of $\mathbf{k}^\rho$ when the subjective discount rate r is given.

From the definition above, the NMF is a set of $(\mathbf{x}, \mathbf{z})$ capital stock vectors which arise from the exact same net benefit as that of OSS when it is evaluated by the prices of OSS. Also, the VMF is the projection of a flat on the surface of the utility function V that is supported by the price vector $(-\mathbf{p}^\rho,\ \rho\mathbf{p}^\rho,\ 1)$ onto the $(\mathbf{x}, \mathbf{z})$-space. In Takahashi (1985), I consider the case of the objective function where n capital goods as well as pure-consumption goods are also consumable. Here, the capital goods are not consumable but the discounted sum of the sequence of pure-consumption goods is directly evaluated. Due to the well-established Nonsubstitution Theorem, along the OSS, a unique technology matrix $\mathbf{A}^\rho$ defined before will be chosen.

By exploiting this fact, we will re-characterize the VMF as a more tractable formula with the (n+1) by (n+1) matrix $\mathbf{A}^\rho$ and (n+1)-dimensional vectors as follows:

Lemma 6. When $\mathbf{A}^\rho$ is non-singular, $(\mathbf{x}, \mathbf{z}) \in \mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$ if and only if there exists $\hat{\mathbf{y}} \equiv (c, \mathbf{y})' = 0$ such that

$$\begin{aligned} i)\ \widehat{\mathbf{x}} &= \mathbf{A}^\rho\widehat{\mathbf{y}} \\ ii)\ \widehat{\mathbf{z}} &= \left(\frac{1}{1+n}\right)\overline{\mathbf{G}}^{-1}[\widehat{\mathbf{y}} + (\mathbf{I} - \overline{\Delta})]\widehat{\mathbf{x}} \end{aligned}$$

$$\text{where } \widehat{\mathbf{x}} = (1, \mathbf{x}), \widehat{\mathbf{z}} = (\mathbf{1}, \mathbf{z}), \overline{\mathbf{G}}^{-1} = \begin{pmatrix} 1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \frac{1}{(1+a_1)} & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \frac{1}{(1+a_n)} \end{pmatrix},$$

$$\overline{\Delta} = \begin{pmatrix} 0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \delta_1 & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \delta_n \end{pmatrix}$$ and $\mathbf{I}$ is a $(n+1)$-dimensional unit matrix.

Proof. From the definition of the NMF, we have the following supporting relation:

$$c + \rho \mathbf{p}^\rho \mathbf{z} - \mathbf{p}^\rho \mathbf{x} = c^\rho + \rho \mathbf{p}^\rho \mathbf{k}^\rho - \mathbf{p}^\rho \mathbf{k}^\rho$$

where c and $\mathbf{y}$ correspond to $(\mathbf{x}, \mathbf{z})$. Furthermore, from the fact that $\mathbf{V}_x(\mathbf{k}^\rho, \mathbf{k}^\rho) = \mathbf{w}^\rho + \mathbf{p}^\rho(\mathbf{I} - \Delta)$ and $\mathbf{V}_z(\mathbf{k}^\rho, \mathbf{k}^\rho) = -(1+g)\mathbf{G}\mathbf{p}^\rho$, the Euler equation implies that $\mathbf{w}^\rho = \mathbf{p}^\rho[\rho^{-1}(1+g)\mathbf{G} - (\mathbf{I} - \Delta)]$. Also note that from the accumulation relations, $\mathbf{z} = (1/(1+g))\mathbf{G}^{-1}[\mathbf{y} + (\mathbf{I} - \Delta)\mathbf{x}]$ and $\mathbf{k}^\rho = (1/(1+g))\mathbf{G}^{-1}[\mathbf{y}^\rho + (\mathbf{I} - \Delta)\mathbf{k}^\rho]$. Substituting theses into the above supporting equation, we finally obtain,

$$(c - c^\rho) + \rho \left(\frac{1}{1+g} \right) \mathbf{G}^{-1} \{[\mathbf{p}^\rho \mathbf{y} - \mathbf{w}^\rho \mathbf{x}] - [\mathbf{p}^\rho \mathbf{y}^\rho - \mathbf{w}^\rho \mathbf{k}^\rho]\} = 0. \tag{14}$$

If $\{[\mathbf{p}^\rho \mathbf{y} - \mathbf{w}^\rho \mathbf{x}] - [\mathbf{p}^\rho \mathbf{y}^\rho - \mathbf{w}^\rho \mathbf{k}^\rho]\} \neq 0$, then the second term turns out to be a vector. This implies that the above equality never holds. Therefore it follows that $\{[\mathbf{p}^\rho \mathbf{y} - \mathbf{w}^\rho \mathbf{x}] - [\mathbf{p}^\rho \mathbf{y}^\rho - \mathbf{w}^\rho \mathbf{k}^\rho]\} = 0$ and $c = c^\rho$. Thus it finally yields that

$$c + \mathbf{p}^\rho \mathbf{y} - \mathbf{w}^\rho \mathbf{x} = c^\rho + \mathbf{p}^\rho \mathbf{y}^\rho - \mathbf{w}^\rho \mathbf{k}^\rho.$$

This result is Condition i), which implies that $(c_0, \mathbf{y}, \mathbf{x})$ should lie on the production frontier of $T(\mathbf{y}, \mathbf{k})$ and in each sector, the chosen technology must be the same as that in the OSS. In other words, the OSS technology matrix $\mathbf{A}^\rho$ will be chosen. Thus on the NMF, the exact same technology matrix as the corresponding OSS is chosen. In other words, given OSS technology matrix $\mathbf{A}^\rho$, the cost minimization and the full-employment conditions for labor and capital goods are satisfied. Therefor, the following equations must hold:

$$1) q^\rho = w_0^\rho a_{00}^\rho + \mathbf{w}^\rho \mathbf{a}_{\cdot 0}^\rho,$$

$$2)\ \mathbf{p}^\rho = w_0^\rho a_{0\cdot}^\rho + \mathbf{w}^\rho \mathbf{a}^\rho,$$

$$3) 1 = a_{00}^\rho c + \mathbf{a}_{0\cdot}^\rho \mathbf{y},$$

$$4)\ \mathbf{x} = a_{\cdot 0}^\rho c + \mathbf{a}^\rho \mathbf{y}$$

The cost-minimization conditions 1) and 2) imply that the same technology as that of OSS is chosen. 3) and 4) means that, under the chosen technology, the full employment conditions hold. It is not difficult to see that 3) and 4) can be summarized as Condition ii). From these conditions, it follows that $c(t)>0$ and $\mathbf{y}(t)\gg\mathbf{0}$ for all t, respectively. Condition ii) are the (n+1)-dimensional capital accumulation equations and $\mathbf{z}$ is determined through this relation.

Note that the dynamics of the NMF is expressed by the accumulation equation of Condition ii). We will rewrite it using the element of the inverse matrix of $\mathbf{A}^{\rho}$ as follows: first note that $\mathbf{b}^{\rho}=[\mathbf{a}^{\rho}-(1/a^{\rho}_{00})\mathbf{a}^{\rho}_{\cdot 0}\mathbf{a}^{\rho}_{0\cdot}]^{-1}$. Solving Condition ii) with respect to $\mathbf{y}$, we will obtain

$$\mathbf{y}=\mathbf{b}^{\rho}\mathbf{x}+\mathbf{b}^{\rho}_{\cdot 0}.$$

Substituting this into the accumulation equation ii) and solving it with respect to $\mathbf{z}$ yields

$$\mathbf{z}=(1/(1+g))(\mathbf{b}^{\rho}+\mathbf{I}-\Delta)\mathbf{x}-((1/(1+g))\mathbf{b}^{\rho}_{\cdot 0}.$$

Defining $\boldsymbol{\eta}(t)=(\mathbf{x}-\mathbf{k}^{\rho})$ and $\boldsymbol{\eta}(t+1)=(\mathbf{z}-\mathbf{k}^{\rho})$, we will finally obtain the following difference equations, which show the dynamics of the NMF:

$$\boldsymbol{\eta}(t+1)=(\frac{1}{1+g})[(\mathbf{b}^{\rho})^{-1}+\mathbf{I}-\Delta]\boldsymbol{\eta}(t). \tag{15}$$

It is important to notice that the dimension of the NMF could be zero. The following lemma will give us an exact order of its dimension.

Lemma 7. dim $\mathbf{F}(\mathbf{k}^{\rho},\mathbf{k}^{\rho})=n$ and $\mathbf{F}(\mathbf{k}^{\rho},\mathbf{k}^{\rho})\subset int\, D$.

Proof. Let us define $\mathbf{d}=(d_0, d_1, \ldots, d_n)^t\in\mathbf{R}^{n+1}_{+}$ such that $\Sigma^{n}_{i=0}\, a_{0i}d_i=0$ holds. Because the linear constraint must be satisfied, we can exactly choose n linearly independent vectors $\mathbf{d}^h$ $(h=1, \ldots, n-1)$. It is clear that $\mathbf{d}^h$ shows a reallocation of fixed labor among sectors. Moreover, define the following: for $h=1, 2, \ldots, n$ and a positive scalar ε_h,

$$\begin{aligned}\widehat{\mathbf{y}}^h &\equiv \widehat{\mathbf{y}}^{\rho}+\varepsilon_h\mathbf{d}^h,\\ \widehat{\mathbf{x}}^h &\equiv \mathbf{A}^{\rho}\widehat{\mathbf{y}}^h=\mathbf{A}^{\rho}\widehat{\mathbf{y}}^{\rho}+\varepsilon_h\mathbf{A}^{\rho}\mathbf{d}^h\\ &= \widehat{\mathbf{k}}^{\rho}+\varepsilon_h\mathbf{A}^{\rho}\mathbf{d}^h\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathbf{z}}^h &\equiv \overline{\mathbf{G}}^{-1}[\widehat{\mathbf{y}}^h+(\mathbf{I}-\overline{\Delta})\widehat{\mathbf{x}}^h]\\ &= \overline{\mathbf{G}}^{-1}[\widehat{\mathbf{y}}^{\rho}+(\mathbf{I}-\overline{\Delta})\widehat{\mathbf{x}}^{\rho}]+\varepsilon_h\overline{\mathbf{G}}^{-1}[\mathbf{d}^h+(\mathbf{I}-\overline{\Delta})\mathbf{A}^{\rho}\mathbf{d}^h]\\ &= \widehat{\mathbf{k}}^{\rho}+\varepsilon_h\overline{\mathbf{G}}^{-1}[\mathbf{I}+(\mathbf{I}-\overline{\Delta})\mathbf{A}^{\rho}]\mathbf{d}^h\end{aligned}$$

Note that the first element of the vector $\mathbf{A}^{\rho}\mathbf{d}^{h}$ is zero due to the fact that $\Sigma_{i=0}^{n} a_{0i}d_i^h=0$ for all h. Since the first element of $\hat{\mathbf{k}}^{\rho}$ is one, the first element of $\hat{\mathbf{x}}^{h}$ will be one. So the vectors $\hat{\mathbf{x}}^{h}$ $(h=1, \ldots, n)$ are well defined. Since the first element of $\tilde{\mathbf{k}}^{\rho}$ is 1, $\tilde{\mathbf{z}}^{h}$ is also well defined for all h. due to the fact that $\hat{\mathbf{y}}^{\rho} \gg \mathbf{0}$ and $\hat{\mathbf{k}}^{\rho} \gg \mathbf{0}$, ε_h can be chosen so that $\hat{\mathbf{y}}^{h} > \mathbf{0}$, $\hat{\mathbf{x}}^{h} > \mathbf{0}$ and $\hat{\mathbf{z}}^{h} > \mathbf{0}$ for all h. From our way of construction, the vectors $\hat{\mathbf{y}}^{h}$, $\hat{\mathbf{x}}^{h}$ and $\hat{\mathbf{z}}^{h}$ satisfy Lemma 2 and the corresponding vector $(\mathbf{x}^{h}, \mathbf{z}^{h})$ also belongs to $\mathbf{F}(\mathbf{k}^{\rho}, \mathbf{k}^{\rho})$ for all h. This implies that there are n linearly independent vectors $(\mathbf{x}^{h}-\mathbf{k}^{\rho}, \mathbf{z}^{h}-\mathbf{k}^{\rho})$. Therefore, there are exactly n linearly independent line segments on the NMF, $\mathbf{F}(\mathbf{k}^{\rho}, \mathbf{k}^{\rho})$. In other words, the NMF has an n-dimensional facet (flat) containing the OSS, $(\mathbf{k}^{\rho}, \mathbf{k}^{\rho})$. This completes the proof.

Using Lemma 7, we will show Lemma8.

Lemma 8. $\mathbf{F}(\mathbf{k}^{\rho}, \mathbf{k}^{\rho})$ is a continuous correspondence of $\rho \in [\bar{\rho}, 1)$.

Proof. See the Appendix.

We need the following definition:

Definition. The NMF is *stable* if there are no cyclic paths on it.

The stability of the NMF takes very important roles in demonstrating the Turnpike properties as we will see soon. Under the both GCI conditions, we actually show that the NMF is stable as we will demonstrate in the next section. Due to the continuity of the NMF, if the stability of the NNF with $\rho=1$ would have been proved, McKenzie's Neighborhood Turnpike Theorem could be applicable as shown in Takahashi (1985) and Takahashi (1992), and finally we would demonstrate the following theorem:

Theorem 1. (Neighborhood Turnpike Theorem) *Provided that the VMF is stable. Then for any* $\varepsilon>0$, *there exists a* $\bar{\rho}>0$ *such that for* $\rho \in [\bar{\rho}, 1)$ and the corresponding $\varepsilon(\rho)$, *any optimal path* $\{\mathbf{k}_t^{\rho}\}^{\infty}$ *with a sufficient initial capital stock* $\mathbf{k}(0)$ *eventually lies in the* ε*-neighborhood of* $\mathbf{k}^{\rho}$. Furthermore, as $\rho \to \infty$, $\varepsilon(\rho) \to 0$.

Proof. See the argument of Section 4 of Takahashi (1993).

The Neighborhood Turnpike Theorem means that any optimal path must be trapped in a neighborhood of the corresponding OSS and the neighborhood can be taken as small as possible by making ρ close enough to one.

5 Turnpike Theorem

The full Turnpike Theorem is described as the following theorem:

Full Turnpike Theorem There is a $\bar{\rho}>0$ close enoug to 1 such that for any $\rho \in [\bar{\rho}, 1)$, an optimal path $\mathbf{k}^{\rho}(t)$ with the sufficient initial capital stock will asymptotically converge to the optimal steady state $\mathbf{k}^{\rho}$.

As we have shown, under Assumption 7, the dimension of the VMF is n. We will keep this assumption henceforth. On the other hand, the dynamics of the VMF is expressed by the n-dimensional linear difference equation (8). To show the full turnpike theorem we need to strengthen the generalized capitl intensity conditionds, GCI-I and GCI-II.

Remark 3 *the first to be noted that in the efficiency unit term, the full turnpike means that each sector's optimal path converges to the optimal steady state. In original terms of series, any industry's per-capita capital stock and output grow at the rate of industry's TFP. Thus our original purpose will be accomplished by showing the Full Turnpike Theorem.*

We use the following property to show the stability of the VMF.

Lemma 9. Let us consider the following difference equation system with the equilibrium $\mathbf{x}_e = 0$,

$$\mathbf{x}(t+1) = (\mathbf{C} + \mathbf{I})\mathbf{x}(t),$$

where $\mathbf{x}(t) \in R^n$ and $\mathbf{C}$ is an $n \times n$ matrix. If $\mathbf{C}$ has the negative d.d. for rows, $\mathbf{C}+\mathbf{I}$ is a contraction for $\mathbf{x}(t) \neq 0$ with the maximum norm $\|\cdot\|$, i.e., and the equation system is globally asymptotically stable and the Liapunov function is $\mathbf{V}(\mathbf{x}) = \|\mathbf{x}\|$, where $\|\cdot\|$, is defined as $\|\mathbf{x}\| = \max_i c_i \, |x_i|$ and c_i is a given set of positive numbers. Furthermore, if $\mathbf{C}$ has the positive d.d. for rows, $\mathbf{C}+\mathbf{I}$ exhibits total explosiveness for $\mathbf{x}(t) \neq 0$.

Proof. The first part comes from the result in Neuman (1961, pp.27-29). On the contrary, if $\mathbf{C}$ has the negative q.d.d. for rows, $\mathbf{C}+\mathbf{I}$ has eigenvalues with their absolute values greater than one. This comes from the fact that if $\mathbf{C}$ has the positive d.d. for rows, then its eigenvalues have a positive real part. Thus the system is explosive; any path will diverge from equilibrium.

We may use the second property later. We will first prove the following theorem:

Lemma 10. Under the negative (positive) d.d., the n-dimensional NMF, $\mathbf{F}(\mathbf{k}^\rho, \mathbf{k}^\rho)$ where $\rho \in [\bar{\rho}, 1)$ turns out to be a linear stable (unstable) manifold.

Proof. Because $\mathbf{b}^\rho + \mathbf{I} - \Delta = [\mathbf{b}^\rho + (\mathbf{I} - \Delta) - (1+g)\mathbf{G}] + (1+n)\mathbf{G}$, it follows that $(1/(1+g)\mathbf{G}[\mathbf{b}^\rho + \mathbf{I} - \Delta] = (1/(1+g))\mathbf{G}[\mathbf{b}^\rho + (\mathbf{I} - \Delta) - (1+n)\mathbf{G}] + \mathbf{I}$. Defining $\mathbf{C} = (1/(1+g))\mathbf{G}[\mathbf{b}^\rho + (\mathbf{I} - \Delta) - (1+g)\mathbf{G}]$, Eq.(8) can be rewritten as:

$$\boldsymbol{\eta}(t+1) = (\mathbf{C} + \mathbf{I})\boldsymbol{\eta}(t).$$

Note again that $\boldsymbol{\eta}(t) = (\mathbf{x} - \mathbf{k}^\rho)$ and $\boldsymbol{\eta}(t+1) = (\mathbf{z} - \mathbf{k}^\rho)$. Thus applying Lemma 4, under the negative d.d. (the positive d.d.), any path on NMF will converge to (diverge from) the OSS.

From this lemma, under the Strong GCI-II condition, the local stability and the stability of the NMF hold simultaneously. The stability of the NMF implies that the Neighborhood turnpike holds. Thus combining both results, the following full Turnpike theorem will be established.

Corollary. Under the Strong GCI-II condition, the full Turnpike Theorem will be established.

Proof. To achieve the full Turnpike theorem, we need to combin the Neighborhood Turnpike Theorem and the local stability of the OSS. The Neighborhood Turnpike Theorem means that any optimal path should be trapped in the neighborhood of the OSS. Thus if the local stability holds in the neighborhood of OSS, then the optimal path must jump on the stable manifold, here the NMF itself, and will converge to the OSS. Thus the full Turnpike theorem will be established.

On the other hand, to show the local stability under the GCI-I condition, we need to utilize the following well-known lemma by Levhari and Liviatan (1972):

Lemma 11. Provided that det $\mathbf{V}^{\rho}_{xz} \neq \mathbf{0}$, if the following characteristic equation, given by expanding the Euler equation around the OSS, has λ as a root, then it also has $1/(\rho\lambda)$,

$$\left|\mathbf{V}^{\rho}_{xz}\lambda^2 + (\mathbf{V}^{\rho}_{xx} + \mathbf{V}^{\rho}_{zz})\lambda + \mathbf{V}^{\rho}_{zx}\right| = 0. \tag{16}$$

Proof. See Levhari and Liviatan (1972).

Lemma 12. Under the GCI-I condition, the OSS satisfies the local stability.

Proof. All we need to show is that det $\mathbf{V}^{\rho}_{xz} \neq \mathbf{0}$ under the GCI-I condition due to Lemma 11. From the fact that $\mathbf{V}_x(\mathbf{k}^{\rho}, \mathbf{k}^{\rho}) = \mathbf{p}^{\rho}(\mathbf{I} - \Delta) + \mathbf{w}^{\rho}$ (see Benhabib and Nishimura (1985) for a two-sector case) and Lemma 4, we may show that

$$\mathbf{V}^{\rho}_{xz} = -(\mathbf{b}^{\rho})^{-1}[\mathbf{b}^{\rho} + (\mathbf{I} - \Delta)]\mathbf{T}^{\rho}_{22}(\mathbf{b}^{\rho})^{-1}$$

where $T^{\rho}_{22} = [\partial^2 T(\mathbf{y}^{\rho}, \mathbf{k}^{\rho})/\partial \mathbf{k}^2]$. Then,

$$\det \mathbf{V}^{\rho}_{xz} = -(\det(\mathbf{b}^{\rho})^{-1})^2 \det[\mathbf{b}^{\rho} + (\mathbf{I} - \Delta)] \det \mathbf{T}^{\rho}_{22}$$

Since $\mathbf{T}^{\rho}_{22}$ is negative-definite, it is non-singular. Furthermore, $[\mathbf{b}^{\rho} + (\mathbf{I} - \Delta)]$ has a quasi-dominant diagonal that is positive under the GCI-I condition, it also nonsingular. Thus $\mathbf{V}^{\rho}_{xz}$ is non-singular. On the other hand, the GCI-I condition implies that the VMF is explosive. This means that there are n characteristic roots with absolute value greater than one. Applying Lemma 11, this also implies that there are n characteristic roots with its absolute value less than one. So the OSS satisfies the local stability.

Thus we have established the following theorem:

Theorem 2 Under the both GCI conditions, the OSS $\mathbf{k}^{\rho}$ exhibits the full Turnpike Theorem.

Proof. Under the GCI-II condition, the full Turnpike Theorem will be established due to the above corollary. On the other hand, under the GCI-I condition, from Lemma 12, the OSS will exhibit the local stability. Since any path on the VMF is totally unstabl, the NMF is "stable" and the Neighborhood Turnpike Theorem hold. Combining both results again, the full

Turnpike Theorem is also established. This completes the proof.

6 ConcludingRemarks

We have demonstrated the Turnpike properties under the two types of the generalized capital intensity conditions. As I mentioned before, the full turnpike means that each sector's per-capita capital stock and output converge to the industry's own steady state paths respectively. For example, the per-capita capital stock of the agriculture sector grows at its own rate of technical progress along its steady state, but another sector, say a manufacturing sector also grows at its own rate of technical progress along its own steady state. The similar explanation can be applicable to other sectors. Thus our theoretical result derived above is consistent with the evidence obtained in recent empirical research.

References

Benhabib, J. and K. Nishimura, 1979, "On the uniqueness of steady state in an economy with heterogeneous capital goods," *International Economic Review Vol.*20, 59-81.

Benhabib, J. and K. Nishimura, 1981, "Stability of equilibrium in dynamic models of capital theory," *International Economic Review* 22, 275-293.

Benhabib, J. and K. Nishimura, 1985, "Competitive equilibrium cycles," *Journal of Economic Theory* 35, 284-306.

Benhabib, J. and A. Rustichini, 1990, "Equilibrium cycling with small discounting," *Journal of Economic Theory* 52, 423-432.

Burmeister, E. and A. Dobell, 1970. *Mathematical Theory of Economic Growth* (Macmillan, London).

Burmeister, E. and D. Grahm, 1975, "Price expectations and global stability in economic systems," *Automatica* 11, 487-497.

Gantmacher, F., 1960. *The Theory of Matrices Vol.*1 (Chelsea, New York).

Inada, K., 1971, "The production coefficient matrix and the Stolper-Samuelson condition," *Econometrica* 39, 88-93.

Jones, R., S. Marjit and T. Mitra, 1993, "The Stolper-Samuelson theorem: Links to dominant diagonals," in: R. Becker, M. Boldrin, R. Jones and W. Thomson, eds., *General Equilibrium, Growth and Trade II-the legacy of Lionel McKenzie*, (Academic Press, San Diego).

Levhari, D. and N. Liviatan, 1972, "On stability in the saddle-point sense," *Journal of Economic Theory* 4, 88-93.

Lucas, R., 1988, "A Mechanics of Economic Development," *Journal of Monetary Economics* 22, 3-42.

Mangasarian, O., 1966, "Sufficient conditions for the optimal control of nonlinear systems," *Journal of SIAM Control* 4, 139-152.

McKenzie,L., 1960. "Matrices with dominant diagonal and economic theory," in: K. Arrow, S. Karin and P. Suppes, eds., *Mathematical Methods in the Social Sciences*, (Stanford University Press).

McKenzie, L., 1983. "Turnpike theory, discounted utility, and the von Neumann facet," *Journal of Economic Theory* 30, 330-352.

McKenzie, L., 1984, "Optimal economic growth and turnpike theorems," in: K. Arrow and M. Intriligator, eds., *Handbook of Mathematical Economics* Vol.3, (North-Holland, New York).

Murata, Y., 1977. *Mathematics for Stability and Optimization of economic Systems* (Academic Press, New York).

Neuman, P., 1961. Approaches to stability analysis. *Economica* 28, 12-29.

OECD, 2003., *The Sources of Economic Growth in OECD Countries*.

Romer, P., 1986. Increasing Returns and Lomg-run growth. *Journal of Political Economy* 94, 1002-1037.

Samuelson, P., 1945, *Foundations of Economic Analyssis* (Harvard University Press).

Scheinkman, J., 1976, "An optimal steady state of n-sector growth model when utility is discounted," *Journal of Economic Theory* 12, 11-20.

Srinivasan, T., 1964, "Optimal savings in a two-sector model of growth," *Econometrica* 32, 358-373.

Takahashi, H., 1985, *Characterizations of Optimal Programs in Infinite Economies*, Ph.D. Dissertation, the University of Rochester.

Takahashi, H., 1992, "The von Neumann facet and a global asymptotic stability," *Annals of Operations Research* 37, 273-282.

Takahashi, H., 2001, "A stable optimal cycle with small discounting in a twosector discrete-time model," *Japanese Economic Review* 52, No. 3, 328-338.

Takahashi, H., K. Mashiyama, T. Sakagami, 2009, "Why did Japan grow so fast during 1955-1973 and the Era of High-speed Growth end after the Oil-shock?: Measuring capital intensity in the postwar Japanese economy," Working Paper 08-2, Dept. of Economics, Meiji Gakuin University.

Yano, M., 1990, "Von Neumann facets and the dynamic stability of perfect fore-sight equilibrium paths in Neo-classical trade models," *Journal of Economics* 51, 27-69.

APPENDIX

We will prove here Lemma 8.

Proof. Since the VMF is upper-semi continuous from its definition, all I need to establish is that $\mathbf{F}(\mathbf{k}^{\rho}, \mathbf{k}^{\rho}) = \mathbf{F}(\mathbf{k}^{\rho})$ is lower-semi continuous (l.s.c.) at $\rho \in [\bar{\rho}, 1)$. Under Assumption 7, we can choose n labor redistribution vectors $\mathbf{d}^h$ as follows:

$$\mathbf{d}^h = (-1, -1, -1, \cdots, -1, \sum_{i=0}^{n} a_{0i}^{\rho}/a_{0h}^{\rho}, -1, \cdots, -1)\ (h = 0, 1, \cdots, n).$$

This means that each producing sector transfers one unit of labor to the h^{th} sector. From Lemma 4, $\mathbf{d}^h$ is a continuous vector function of ρ in $\rho \in [\bar{\rho}, 1)$ and that $\Sigma_{i=0}^{n} a_{0i}^{\rho} d_i^h = 0$ for all h. This means that there are n linearly independent redistribution vector $\mathbf{d}^h$. Let us denote these redestribution vector as $\mathbf{d}^h$ $(h=1, \ldots, n)$. Henceforth, we may use the notation, $\mathbf{k}(\rho)$, $\mathbf{d}^h(\rho)$ (h=1, ..., n) and $\mathbf{F}(\mathbf{k}(\rho))$ for denoting OSS, redistribution vectors and the VMF respectively. Using $\mathbf{d}^h(\rho)$ and from Lemma 6, we can define the following n linearly independent vectors for each h $(h=1, \ldots, n)$: $\hat{\mathbf{x}}^h(\rho) = \mathbf{k}(\rho) + \varepsilon_h \mathbf{A}(\rho)\mathbf{d}^h(\rho)$ and $\hat{\mathbf{z}}^h(\rho) = \mathbf{k}(\rho) + \varepsilon_h \bar{\mathbf{G}}^{-1}[\mathbf{I} + (\mathbf{I} - \bar{\Delta})\mathbf{A}(\rho)]\mathbf{d}^h(\rho)$ where ε_h is chosen so that $\hat{\mathbf{x}}^h(\rho) \gg \mathbf{0}$ and $\hat{\mathbf{z}}^h(\rho) \gg \mathbf{0}$ $(h=1, \ldots, n)$. Let us arbitrarily choose a point $(\mathbf{x}, \mathbf{z}) \in \mathbf{F}(\mathbf{k}(\rho))$. Furthermore choose $(\mathbf{x}', \mathbf{z}') \in \mathbf{F}(\mathbf{k}(\rho'))$ where $(\mathbf{x}', \mathbf{z}') \in int\, D$, $(\mathbf{x}, \mathbf{z}) \neq (\mathbf{x}', \mathbf{z}')$ and $\rho' \in [\bar{\rho}, 1)$ is chosen close enough to ρ. Now let us define the plain $\mathbf{H}^\alpha$ is defined as follows:

$$\mathbf{H}^\alpha \equiv \{(\mathbf{x},\mathbf{z}) \in D : t_0[\alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho), \alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho)] +$$

$$\sum_{h=1}^{n} t_h[\alpha\mathbf{x}^j(\rho') + (1-\alpha)\mathbf{x}^j(\rho), \alpha\mathbf{z}^j(\rho') + (1-\alpha)\mathbf{z}^j(\rho)]\} \; where \; \sum_{h=1}^{n} t_h = 1.$$

We can always find an intersection $(\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ between $\mathbf{H}^\alpha$ and the line obtained by connecting points $(\mathbf{x}, \mathbf{z})$ and $(\mathbf{x}', \mathbf{z}')$ unless $(\mathbf{x}, \mathbf{z}) = (\mathbf{x}', \mathbf{z}')$. Since $(\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ is on the plain $\mathbf{H}^\alpha$, it can also be expressed as follows:

$$(\mathbf{x}^\alpha, \mathbf{z}^\alpha) = t_0^\alpha[\alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho), \alpha\mathbf{k}(\rho') + (1{-}\alpha)\mathbf{k}(\rho)] +$$

$$\sum_{h=1}^{n} t_h^\alpha[\alpha\mathbf{x}^j(\rho') + (1-\alpha)\mathbf{x}^j(\rho), \alpha\mathbf{z}^j(\rho') + (1-\alpha)\mathbf{z}^j(\rho)] \; where \; \sum_{h=1}^{n} t_h^\alpha = 1.$$

where $\alpha \to 0$, $[\alpha\mathbf{k}(\rho') + (1-\alpha)\mathbf{k}(\rho)] \to \mathbf{k}(\rho)$ and $[\alpha\mathbf{x}^h(\rho') + (1-\alpha)\mathbf{x}^h(\rho), \alpha\mathbf{z}^h(\rho') + (1-\alpha)\mathbf{z}^h(\rho)] \to (\mathbf{x}^\rho, \mathbf{z}^\rho)$ for $h = 1, \ldots, n$. Therefore $(\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ converges to $(\mathbf{x}, \mathbf{z})$ as $\alpha \to 0$. Also note that $(\mathbf{x}^\alpha, \mathbf{z}^\alpha) \in int\, D$ due to the convexity of D and the fact that $(\mathbf{x}', \mathbf{z}') \in int\, D$. On the other hand, because of the continuity of $\mathbf{k}(\rho)$, $\mathbf{x}(\rho)$ and $\mathbf{z}(\rho)$ in $\rho \in [\bar{\rho}, 1)$. for any $\varepsilon^\alpha > 0$ there exists $\delta^\alpha > 0$ such that $|\rho^\alpha - \rho| < \delta^\alpha$ implies that

$$\|\mathbf{k}(\rho^\alpha) - [\alpha\mathbf{k}(\rho') + (1-\alpha)\mathbf{k}(\rho)]\| < \varepsilon^\alpha$$

and for $h = 1, \ldots, n$,

$$\left\|(\mathbf{k}(\rho^\alpha), \mathbf{k}(\rho^\alpha)) - [\alpha\mathbf{x}^h(\rho') + (1-\alpha)\mathbf{x}^h(\rho), \alpha\mathbf{z}^h(\rho') + (1-\alpha)\mathbf{z}^h(\rho)]\right\| < \varepsilon^\alpha$$

where $\rho^\alpha = \alpha\rho' + (1-\alpha)\rho$ and $\|\bullet\|$ is the Euclidean norm. Furthermore as $\alpha \to 0$, $\varepsilon^\alpha \to 0$. Now let us also define a point $(\bar{\mathbf{x}}^\alpha, \bar{\mathbf{z}}^\alpha)$ as follows:

$$(\bar{\mathbf{x}}^\alpha, \bar{\mathbf{z}}^\alpha) = t_0^\alpha(\mathbf{k}(\rho^\alpha), \mathbf{k}(\rho^\alpha)) + \sum_{h=1}^{n} t_h^\alpha(\mathbf{x}^j(\rho^\alpha), \mathbf{z}^j(\rho^\alpha))$$

where $(t_0^\alpha, t_1^\alpha, \ldots, \ldots, t_n^\alpha)$ is used to define $(\bar{\mathbf{x}}^\alpha, \bar{\mathbf{z}}^\alpha)$. By choosing α close enough to zero, we can make $(\bar{\mathbf{x}}^\alpha, \bar{\mathbf{z}}^\alpha)$ close enough to $(\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ and $(\bar{\mathbf{x}}^\alpha, \bar{\mathbf{z}}^\alpha) \in int\, D$. Thus $(\bar{\mathbf{x}}^\alpha, \bar{\mathbf{z}}^\alpha)$ is feasible. Since it is also expressed as the linear combination of $(n-1)$ linearly independent vectors $\mathbf{x}^h$, $\mathbf{y}^h$ and $\mathbf{z}^h$, it follows that $(\bar{\mathbf{x}}^\alpha, \bar{\mathbf{z}}^\alpha) \in \mathbf{F}(\mathbf{k}(\rho^\alpha))$. Also note that due to our way of construction of $(\bar{\mathbf{x}}^\alpha, \bar{\mathbf{z}}^\alpha)$, as $\alpha \to 0$, $(\bar{\mathbf{x}}^\alpha, \bar{\mathbf{z}}^\alpha) \to (\mathbf{x}^\alpha, \mathbf{z}^\alpha)$. Now make α converge to zero, then $(\mathbf{x}^\alpha, \mathbf{z}^\alpha) \to (\mathbf{x}^\alpha, \mathbf{z}^\alpha)$ and $(\mathbf{x}^\alpha, \mathbf{z}^\alpha) \to (\mathbf{x}, \mathbf{z})$. Thus $(\bar{\mathbf{x}}^\alpha, \bar{\mathbf{z}}^\alpha) \in \mathbf{F}(\mathbf{k}(\rho^\alpha))$ converges to $(\mathbf{x}, \mathbf{z}) \in \mathbf{F}(\mathbf{k}(\rho))$. therefore $\mathbf{F}(\mathbf{k}(\rho))$ is l.s.c. at $\rho \in [\bar{\rho}, 1)$. Apply the same arguments to any point of $\mathbf{F}(\mathbf{k}(\rho))$ and any $\mathbf{F}(\mathbf{k}(\rho))$ of $\rho \in [\bar{\rho}, 1)$, it follows that $\mathbf{F}(\mathbf{k}(\rho))$ is l.s.c. on $[\bar{\rho}, 1)$.