The Core Involving Public Goods Revisited:
A Diagrammatic Analysis

Harutaka Takahashi[1]

ABSTRACT

Using a diagram called a "Kolm triangle" adopted in Kolm (1970), the important issues of 1) Pareto efficiency and the core, and 2) Lindahl equilibrium and the core in the resource allocation problem involving public goods analyzed by Foley (1970) and Nikaido (1976) can be illustrated solely using plane figures. One advantage of using such a diagrammatical method is that it allows an intuitive understanding of the core of the problem without using highly complicated mathematics to any degree.

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1. Introduction

Foley (1970) has pointed out and Muench (1972) has shown by an example that the core of a public goods economy does not shrink to the Lindahl equilibrium even when the number of traders becomes very large. The implication of results is important, because it implies that bargaining among many traders need not lead to the Lindahl solution. Nikaido (1976) has tried to explain this more intuitively by using the diagram. Although Nikaido (1976) does not mention the Kolm triangle directly, diagrams similar to those in Kolm (1970) are used for analysis. His arguments and proofs, however, are mainly based on the three dimensional graph.

In this note I will prove the non-shrinkage property of the core of public goods economy

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totally based on the Kolm triangle. In Section 2, I will set up the public goods economy model and will derive the Kolm triangle based on the argument by Nikaido (1976) as a two-dimensional simplex in the three dimensional diagram. The detailed properties of the triangle will be left for Thomson (1999) and Ley (2002). The core will be defined there. In Section 3, I will prove the non-shrinkage of the core by using the Kolm triangle. In Section 4, I will discuss the core and the Lindahl equilibrium in an intuitively comprehensible fashion. I will give some comments in Section 5.

2. The Core and the Kolm Triangle

The model assumed by the Kolm triangle is fundamentally a two-person – two-commodity model just as in the case of the Edgeworth box-diagrams. In other words, the assumption is that two economic agents who are given the initial amount of two commodities exchange the two commodities. The Kolm triangle has two important differences with the box-diagrams. One is that public goods (pure public goods), which have both properties of complete non-excludability and complete non-rivalness, are added to private goods that might be privately owned and consumed by every agent. The other is that public goods are transformed from private goods under certain technical restrictions. Following Nikaido (1976), salient assumptions of the model assumed by the Kolm triangle are listed below.

Assumption 1: Composed of two individuals, A and B.

Assumption 2: Goods of two types, i.e., private goods and completely non-excludable and completely non-rival public goods, exist.

Assumption 3: Each agent owns a certain initial amount in the form of private goods and consumes both private and public goods.

Assumption 4: Private and public goods are technically transformed one-to-one.

Consequently, in the terminology of economics, the marginal rate of transformation (MRT) of private and public goods is fixed at 1. Furthermore, the following symbols are defined for the subsequent argument.

g : the amount of public goods consumed by the two parties

x_i : the amount of private goods consumed by i (i=A, B)

w_i : the initial amount of resources held by i (i=A, B)

Let us denote the private goods consumption of i th person as x_i ($i = A, B$) and the total amount of the endowments as $w\,(= w_A + w_B)$. Since $w = x_A + x_B + g$ holds, we can draw the consumption possibility set as a two-dimensional simplex in the three dimensional diagram, which are spanned x_A, x_B *and* g axes as shown in Fig. 1, which is a similar diagram drawn as

Fig.2 in Nikaido (1976). Note that the two-dimensional simplex ABC [2] is an equilateral triangle in our case. Furthermore, if we look at the simplex from the origin O and draw it as the equilateral triangle, we obtain the Kolm triangle.

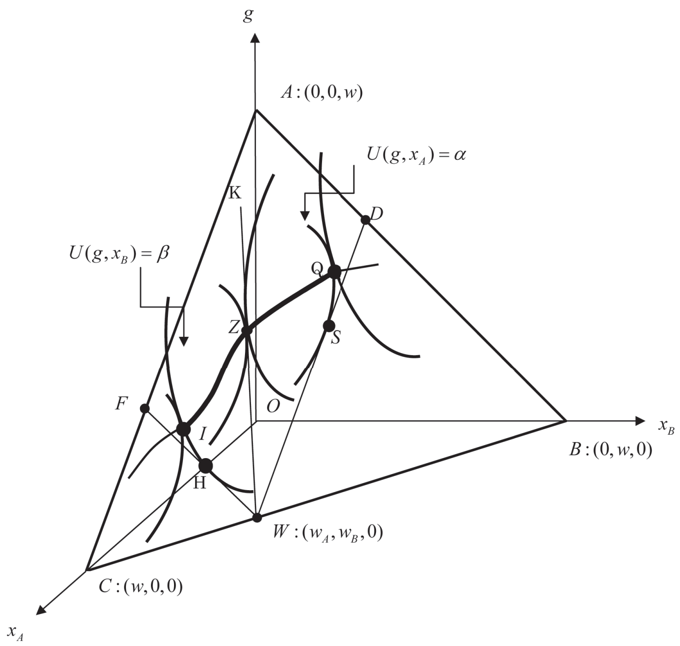

Figure 1. A Version of the Edgeworth Box-diagram involving Public Expenditures

A possible allocation is denoted by a triplicate (x_A, x_B, g). As discussed in detail in Nikaido (1976), the cylindrical indifference surface of both traders intersect the two-dimensional simplex and provide indifference curves on it as shown in Fig. 1. Point W is the initial endowment point for both traders. Furthermore, the person A can increase the public goods along the line WD, which is the transformation line between private and public goods. Similarly, the person B can increase the public goods along the line WF. As discussed by Nikaido (1976), the core of the basic economy is the portion of the contract curve in which the utility levels of traders A and B are at least α, β respectively. The core is the curvilinear segment QI in Fig.1. A Lindahl equilibrium point Z is such a point on the contract curve that the straight line WD is a common tangent of both traders' indifference curves at Z. Looking at this diagram from the origin O, we can draw the Kolm triangle as Fig. 2. More direct construction of the Kolm triangle was discussed in detail by Thomson (1999) and Ley (1996).

2 This triangle was referred to as "a version of the Edgeworth box-diagram involving public expenditures" by Nikaido (1976).

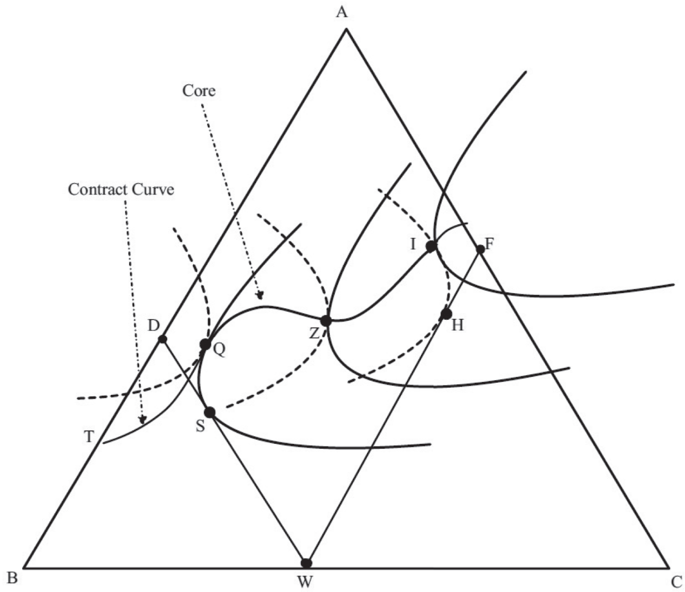

Figure 2. Kolm Triangle

Any allocation point on the contract curve surrounded by the indifference curves tangent at points S and H evidently provides utility that is equal to or greater than the utility gained at the points S and H. Therefore, rather than the state of isolated economy, performing redistribution of some kind between two persons A and B, and moving to an allocation point (e.g. point Z) on a part of the contract curve expressed by QI would result in Pareto improvement. Such an allocation point is an allocation that would not be rejected either by individual A alone, by individual B alone, or by two individuals A and B. As in the case of the box-diagram, a set of such allocation points is called a "**core.**"

Now, let us call an economic agent identical to the individual A "Type-A" and one identical to the individual B "Type-B." In this case, it is known that the core of private goods is reduced as the number of economic agents that are the same types as A and B increases—which becomes the same as competitive equilibrium at the limit (which is called the **limit theorem on the core**). Whether or not the limit theorem holds in an economy that includes public goods is an interesting issue. To analyze this, we designate the economy depicted in Fig. 2 a "**basic economy**" (represented by $\mathscr{E}$) and define as follows the "***n*-fold replicated economy**" (represented by $\mathscr{E}^n$) composed based on the basic economy:

Definition: Resource allocation (g, x_{A1}, x_{A2}, $\cdots$,x_{An}, x_{B1}, $x_{B2}\cdots$, x_{Bn}) is called an ***n-fold replicated economy*** and is expressed as $\mathscr{E}^n$ when it is satisfies the following condition:

$$(*) \qquad g+\sum_{i=1}^{n} x_{Ai}+\sum_{i=1}^{n} x_{Bi}=nw,$$

where x_{Ai} represents the amount of private goods consumed using a Type-A person and x_{Bj} is the amount of private goods consumed by a Type-B person. Additionally, w represents the amount of initial resources.

Based on this definition, when (g^*, x_A, x^*_B) expresses the resource allocation of the basic economy $\mathscr{E}$, the resource allocation of the n-fold replicated economy satisfies the following conditions.

$$g = ng^*,$$

$$x_{Ai} = x^*_A\ (i=1, \cdots, n),$$

and $$x_{Bj} = \mathrm{x}^*_B\ (j=1, \cdots, n).$$

We will draw a diagram of two-fold replicated economy $\mathscr{E}^2$ from the basic economy $\mathscr{E}$ using the Kolm triangle. Fig. 3 is the Kolm triangle expressing the basic economy $\mathscr{E}$ after removing the factors unnecessary in the current explanation from Fig. 2. We assume that point Z indicates the resource allocation (g^*, x^*_A, x^*_B) of the basic economy $\mathscr{E}$. Because n = 2, the line *ZD* expressing the private goods allocation of Type-A is extended to the point *D*' to become twice the original length. Similarly, line *ZF* for Type-B is extended to point F' to become twice as long. Line *ZU* representing the allocation of public goods (g^*) is also extended to point U' to double the length ($2g^*$). Drawing a straight line passing through each of the three points obtained in this way and so that the sides are parallel to each side of the original equilateral triangle yields equilateral triangle △*A*'*B*'*C*'. Based on this drawing method, each side of the equilateral triangle △*A*'*B*'*C*' is twice as long as the sides of the original equilateral triangle △*ABC*. Similarly, using an arbitrary allocation point in the original equilateral triangle

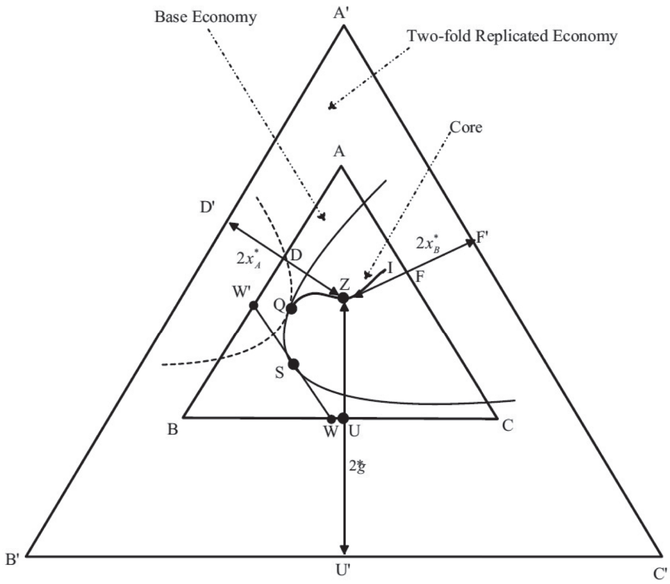

Figure 3. Two-flod Replicated Economy

△ ABC, an equilateral triangle expressing the two-fold replicated economy for the allocation point can be drawn. Extending each allocation n times creates the Kolm triangle of the n-fold replicated economy. In the n-fold replicated economy composed in this way in an economy consisting only of private goods, **coalition** of the individuals is known to improve their economic conditions and to reduce the set of core allocations. As proven by Nikaido (1976), however, the core is not reduced in an economy including public goods, but the set of core allocations remains as it is.

Therefore, QI depicted in the equilateral triangle △ABC in Fig.3 as the set of core allocations of the basic economy $\mathscr{E}$ still represents a core allocation set in the equilateral triangle △$A'B'C'$ that indicates the two-fold replicated economy $\mathscr{E}^2$. The same is generally true for an n-fold replicated economy $\mathscr{E}^n$.

To show this, we will demonstrate, in this case, that a total of three individuals consisting of two Type-A individuals and one Type-B individual could not compose a better allocation for them than point Q expressing the core allocations of the basic economy $\mathscr{E}$ in Fig.3 by forming a coalition. In other words, we will prove that the core allocation point Q in the economy $\mathscr{E}$ would not disappear in the two-fold replicated economy $\mathscr{E}^2$.

A diagram resembling Fig.3 is drawn as Fig.4. First, the line DQP that goes through point Q is drawn parallel to line $W'W$ constituting the budget line of Type-A in an isolated economy. On this straight line, the allocation point F that makes the level of public goods a half of the allocation at the point Q is selected. If the allocation point F provides Type-A individuals with

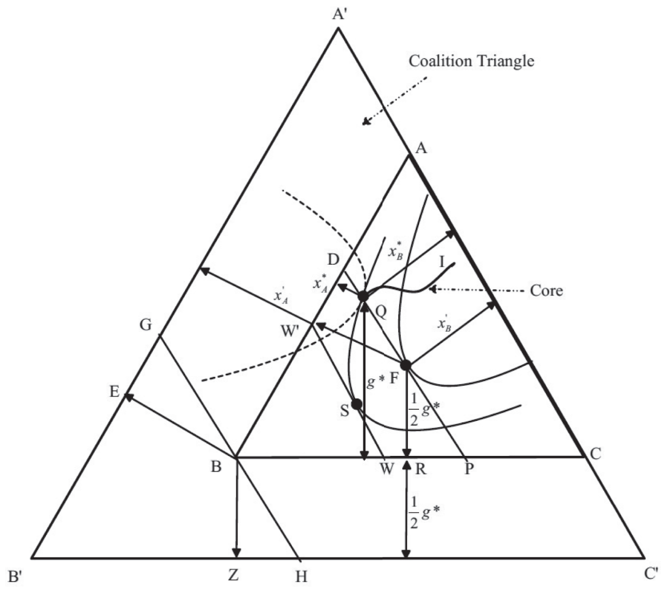

Figure 4. Coalition Triangle

utility that is less than that provided by the allocation point Q, then planning such a collation would be meaningless to begin with. The following argument therefore assumes that the utility of the Type-A individuals at this allocation point F is greater than the utility from allocation point Q. In other words, as depicted in Fig.4, point F on line QP is located inside the area surrounded by the indifference curve that passes through point Q. Therefore, the indifference curve of Type-A giving greater utility than the indifference curve going through the point Q goes through the point F. At the allocation point F, the same private goods as the allocation point Q for the Type-B individual are allocated. We assume now that the two Type-A individuals and one Type-B individual have formed a coalition, excluded the other Type-B individual and are discussing transfer of their welfare level from the allocation point Q to the allocation point F that provides Type-A with great utility. Such a coalition of three individuals can be sought by extending the level of public goods at the point F twice in the perpendicular direction and simultaneously extending the level of private goods of Type-A at the point F twice in the perpendicular direction from the side AB. The triangle depicted in this way becomes an equilateral triangle based on the drawing method. We designate this triangle a **coalition triangle**. In this case, for the Type-B individual participating in the coalition, the same public goods level g^* and private goods level x^*_B at the allocation point Q have been realized at the allocation point F. Accordingly, the utility of the one Type-B individual participating in the coalition is identical to point Q. As noted previously, however, the utility of the two Type-A individuals is greater at point F than at point Q. Consequently, if the allocation point F is feasible in a two-fold replicated economy, then by forming a coalition, the three individuals are able to move to the allocation point F at which the economic conditions of the two Type-A individuals are improved. The core allocation point Q of the basic economy therefore disappears in a two-fold replicated economy. Would such a coalition plan be feasible? For allocation point F to be feasible in a two-fold replicated economy, the following conditions must be satisfied for the implementation.

$$x'_{A1}+x'_{A2}+x'_{B1}+g^* \leq w_{A1}+w_{A2}+w_{B1}$$

Therein, $w_{A1}+w_{B2}$ equals the length of the base of the triangle $\triangle ABC$. The length of the base of the coalition triangle $\triangle ABC$ equals $x'_{A1}+x'_{A2}+x'_{B1}+g^*$. Consequently, when the following conditions are met, the coalition plan proves to be feasible.

Feasibility conditions for the coalition : B'C' ≤ BC + W'W

At this point, the last relation is derived from the fact that the length of $W'W$ equals w_{A2}.

We now assume that the perpendicular lines to the side $A'B'$ and the base of the coalition triangle $\triangle ABC$ created through the coalition of the three individuals from the vertex B of a feasible triangle in the basic economy are named Point E and Point I, respectively. The properties of an equilateral triangle, the triangles $\triangle GBE$ and $\triangle DFW'$ and the triangles $\triangle BZH$ and $\triangle FRP$ are both congruent pairs. Consequently, $GH = DP$ holds. At this stage, $GH > W'W$ holds because of $DP > W'W$. The relation $B'C' > BC + W'W$ is derived from $GH = B'H$ and $B'C' = B'H + HC' = GH + BC$. Therefore, a coalition plan of the three individuals described above is not feasible. The same argument is applicable to an arbitrary point of core allocation. The core allocation QI therefore remains as is in a two-fold replicated economy. The same argument proves to be applicable to a three-fold replicated economy by considering the allocation point for $(1/3)g^*$ on the line QP. For an n-fold replicated economy, too, the fact that none of the core allocation points disappear can be demonstrated by considering the allocation point for $(1/n)g^*$ on the line QP. Consequently, the following important attribute has been proven.

Proposition. When (g^*, x^*_A, x^*_B) ***expresses an arbitrary core allocation of the basic economy*** $\mathscr{E}$ ***, the allocation also constitutes the core allocation of the n-fold replicated economy of the basic economy*** $\mathscr{E}^n$. [3]

Foley (1970) pointed out the following as a general reason why the core is not reduced when public goods are included. When public goods with externalities exist, forming a coalition means that the public goods must be provided to other members of the coalition group. Therefore, an opportunity cost of losing the benefit of public goods that would have been provided by individuals who did not participate in the coalition would be incurred. Therefore, the larger the number of the individuals to be excluded, the greater this opportunity cost and the greater the benefit of the coalition to be lost. As a result, an increase in the number of the individuals to join the coalition would occur, with various allocation points remaining as the core.

3. The Lindahl Equilibrium and the Core

In the argument of core allocation for private goods, the core allocation is known to converge to competitive equilibrium by repeating a replicated economy. As proven in Section 3, however, such contraction of the core does not occur in an economy including public goods. **Lindahl equilibrium** is known as one means to achieve the core allocation discussed in Section 2, which can also be analyzed using the Kolm triangle.[4] The fundamental concept of Lindahl equilibrium is that the government notifies the ratios of contribution of persons A and

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3 This is the theorem proven on page 78 of Nikaido (1976).

4 Refer to the classical study of Foley (1970) for a general analysis of Pareto efficiency and the core and Lindahl equilibrium.

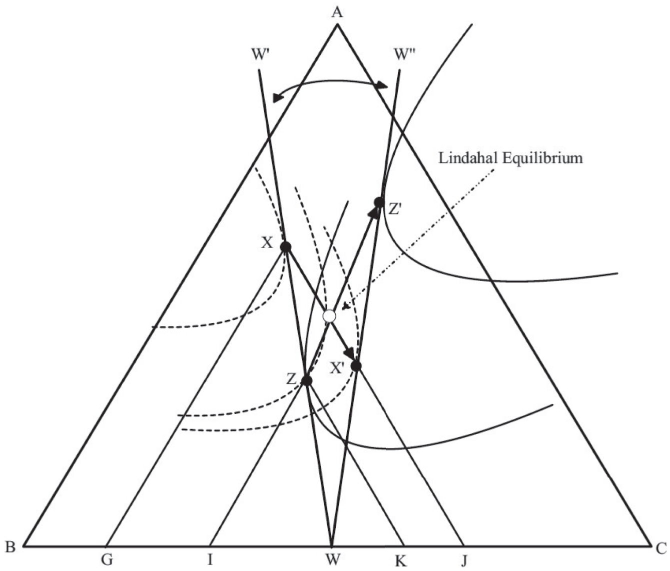

Figure 5. Lindal Equilibrium

B to public goods. Based on those ratios, the government has them report the optimal level of public goods. At this time, if the levels of public goods reported using A and B differ, then an adjustment is made to increase the ratio contributed by the party that has reported the higher level of public goods and to decrease that of the party that has reported the lower level of public goods. Through such an adjustment process, this method achieves a Pareto efficient allocation when the levels of public goods reported by the two parties coincidentally match each other. At this point, the sum of the ratios indicating the marginal rates of substitution of A and B calculated as the ratios of contribution of each party to public goods. These ratios depend only on the slope of the common budget line extended from the point W of the initial amount held. In other words, all levels of public goods determined at arbitrary points on this straight line have the same ratio of contribution. We will calculate, for instance, the contribution ratio of the individual A at arbitrary points X and Z on WW. The contribution ratio at point X is GW/GJ and that at point Z is IW/IK. At this time, IW/GW=IZ/GX holds if the triangles △ WXG and △ WZI are used. Furthermore, GJ=GX and IK=IZ hold based on the properties of an equilateral triangle, which are substituted for the above ratios to result in ; IW/GW=IK/GJ also, GW/GJ=IW/IK is proven.

Given a certain ratio of contribution, we assume that the government adjusts the contribution ratio of individuals who prefer a larger amount of public goods by raising it and that of others by lowering it as described earlier. Making such changes to the contribution ratios mean to turn the common budget line extending from the point W in the Kolm triangle clockwise or counterclockwise using point W as the pivot. In this case, because the contribution ratio of each can be regarded as a price of a type for public goods, the contribution ratio is called a "**Lindahl price**." The government's adjustment mechanism that uses Lindahl prices allows the identification of conditions for the amounts of public goods consumption reported by the two parties to match each other, thereby achieving equilibrium. This equilibrium is a **Lindahl equilibrium**. This state of equilibrium can only be achieved in cases such as that of point Z in Fig. 5 because the indifference curves of the two parties are tangent to the same budget line and the optimal levels of public goods of the two also match only in the case of point Z. Figure 5 presents two case scenarios that can aid consideration of such an adjustment process more specifically. When the common budget line is WW', the individual B selects the point X and the individual A selects the point Z. B. Therefore, signaling a preference of a high level of public goods more than A.[5] The government, in this case, adjusts the contribution ratios to make B's ratio higher and A's ratio lower. This turns the common budget line clockwise. We assume that the new common budget line has become WW'' in this way. Next, B selects the point X' and A selects the point Z' . In this case, A prefers a higher level of public goods than B. The government this time turns the common budget line counterclockwise. In other words, it adjusts the contribution ratios to raise A's ratio and lower B's ratio. The optimal allocation of the private and public goods of each change is made through such a process of adjustment by the government. To determine the Lindahl equilibrium uniquely, we make the following assumptions for the utility function of each economic agent by following Nikaido (1976).

$$\begin{cases} A1.\ \text{Utility function } u_i(g, x_i)(i{=}A, B) \text{ has all properties of concave function normally assumed.} \\ A2.\ \text{The following relationship holds for any positive value } \alpha \\ u_i(g, x_i) \geq u_i(g, x) \rightarrow u_i(\alpha g, x_i) \geq u_i(\alpha g, x_i) \end{cases}$$

Based on assumption A2, when the budget line changes using the initial possession point of the individual A as the pivot as presented in Fig. 6, the optimal consumption allocation of the individual changes parallel to the vertical axis representing public goods while maintaining the demand for private goods as constant.[6] This means that, in the Kolm triangle, the optimal allocation point of each moves parallel to each side of the equilateral triangle because of changes in the contribution ratios. As a consequence, Fig. 5 depicts the movement of the optimal allocation point of the individual A attributable to changes in the contribution ratio as the line XX' parallel to the side AC, and the optimal allocation point of the individual B

5 The length from each equilibrium point to the base represents the optimal amount of public goods for each individual.

6 Refer to the explanation on the page 81 of Nikaido (1976) for details.

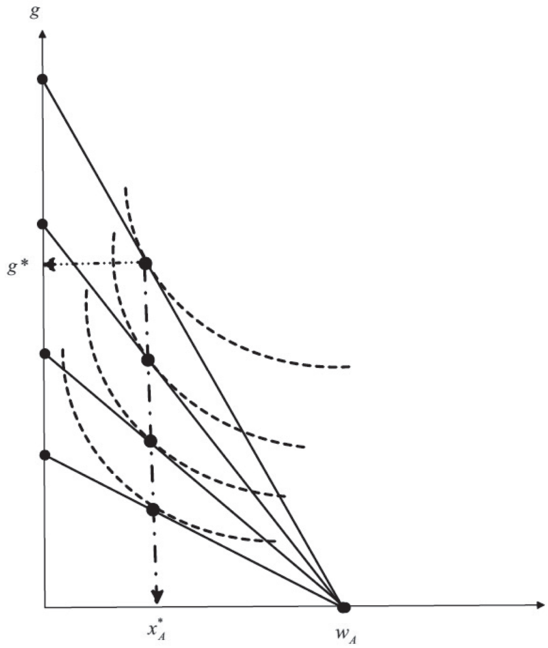

Figure 6. Budget Line and Indifference Curves

as the line ZZ' parallel to the side AB. These two lines apparently intersect mutually only once. According to the preceding description, this intersection becomes Lindahl equilibrium. A unique Lindahl equilibrium has thus been obtained. Furthermore, Lindahl equilibrium is Pareto efficient based on the preceding argument. Lindahl equilibrium is not equilibrium in an isolated economy, which therefore is included naturally in the "core." In addition, the Lindahl equilibrium in the basic economy proves to be Lindahl equilibrium in the n-fold replicated economy based on the proposition proved in Section 2.

4. Conclusion

The argument up to this point has demonstrated that the diagram called the Kolm triangle developed by Kolm (1970) is a useful tool for intuitive understanding of the important issues of "Pareto efficiency and the core" and "Lindahl equilibrium and the core" in the problem of resource allocation involving public goods in the two-person – two-commodity model. As argued by Ley (1996), the Kolm triangle also provides intuitive understanding of issues related to public goods other than this study, which include, for instance, "the problem of voluntary supply of public goods" and "neutrality proposition" discussed by Warr (1983), Bergstrom et al. (1986), and Gradstein et al. (1994).

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