『経済研究』（明治学院大学）第 130 号　2004 年

Marshallian Competitive Economy with Increasing Returns and Free Entry of Firms

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Abstract

A Marshallian general equilibrium model will be constructed and studied. We call our model "Marshallian" because (1) there exist the external increasing returns in Marshall's sense in the economy, (2) there is a continuum of (potential) firms each of which has a technology that allows set-up cost and U-shaped average cost function, and hence (3) firms' free entry and exit occur and the number (mass) of the active firms is determined endogenously according to the profit conditions. We will prove the existence of equilibrium and give a through exposition of its welfare property.

Journal of Economic Literature Classification Numbers: D51, D62, L11.

1 Introduction

In his famous *Principles of Economics* [9], A. Marshall tried to explain the ongoing economic progress by means of increasing returns. It seems that he was aware that "internal" increasing returns generated from convex production functions are not compatible with the competitive behavior of firms. In order to avoid this, he introduced the notion of economies that are external to firms but internal to an industry and claimed that this "external economies" give rise to the increasing returns which are compatible with competitive equilibrium.

However, this idea of increasing returns was seriously attacked by P. Sraffa [17] and independently by F. Knight [8]. They did not believe the claim of Marshall and his followers[1] and argued that competitive equilibrium with increasing returns (no matter whether they are internal or external) was impossible. Knight left his famous remark on the increasing returns that "it is an empty economic box." Although there were indeed the conceptual and technical confusions, the idea had been kept to be used among international trade theorists because of its own charm[2]. Among those, one of the most important paper seems to be that of A. Young [19]. In this paper, he gave an intuitive idea that the increasing returns are driving force of the long-run economic growth.

† This paper is based on chapter 5 of my Ph. D. thesis submitted to University of Rochester. I would like to thank Professors Marcus Berliant and Lionel McKenzie for their helpful comments. Of course the usual caveat applies. Financial support from the Institute for Research in Business and Economics, Meiji Gakuin University is gratefully acknowledged.

1 For example, see F. Graham [5].

2 See Chipman [2] and Helpman [6].

Finally in 1970, the problem was cracked open by J. S. Chipman [3] who gave a clear and simple expression of Marshall's idea and showed that the increasing returns were indeed compatible with competitive equilibrium. In his interpretation, each firm producing commodity $j(=1,\ldots,n)$ has the production function $y_j = k_j z_j$, where y_j is the output and z_j is the input. k_j is treated as a constant by each firm, but actually it is related to the (aggregate level of) input by the condition that $k_j = \kappa_j z_j^{\rho_j - 1}$, where $\rho_j(>1)$ is called the "degree of homogeneity" Then the firm operates subjectively under the constant returns, but the objective production function is "incr easing returns", $y_j = \kappa_j z_j^{\rho_j}$. He assumed that there was only one consumer in the economy with a Cobb-Douglas utility function $\sum_{j=1}^{n} \alpha_j \log x_j$ and one unit of labor, where x_j is the consumption of the final output j. This simplification allowed him to compute the equilibrium directly. Furthermore, he obtained the condition under which the equilibrium to be Pareto optimal. According to his result, the competitive equilibrium is Pareto optimal if $\rho_j = \sum_{k=1}^{n} \alpha_k \rho_k$ for ever $j = 1,\ldots,n$.

P. Romer [13] applied this idea to an optimal growth model and revived A. Young's idea that placed the increasing returns in a central role for the economic growth. He observed that the optimal path in growth models with the increasing returns, if it existed, could grow without bound and the turnpike property did not hold. This is in contrast with the standard growth model. Suzuki [18] proved the existence of equilibrium for a general equilibrium model with the infinite time horizon in which equilibrium path could grow without bound, and confirmed the conjecture of Young and Romer.

In the present paper we will go back to Marshall's original scenario and construct a general equilibrium model with the increasing returns. We call our model "Marshallian" because (1) there exist the increasing returns in Marshall's sense in the economy, (2) there is a continuum of (potential) firms each of which has a technology that allows set up cost and "U-shaped" average cost function, and hence (3) firm's free entry and exit occurs and the number (mass) of the active firms is determined endogeneously according to the profit conditions. The market struture with these properties is thought to be close to that of Marshall [9][3].

The mathematical structure of our model is similar to that of Novshek and Sonnenschein [10], in which they constructed a general equilibrium model with a continuum of firms, each of which is small relative to the whole market (the efficient scale is small) and has a nonconvex production set which allows U-shaped average cost curve. The "convexfying effect of aggregation" brings the convex total production set and existence of competitive equilibrium results. Moreover, they proved the first and second welfare theorems. The present paper adds two more things to their analysis. First, we introduce the increasing returns into their equilibrium existence theorem. This result is thought to be a definitive answer to the criticism of Sraffa and Knight. Second, we give a thorough exposition of the welfare property of the equilibrium. Since the increasing returns arise from externalities, the welfare properties are not trivial matter. We

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3 For the above characterization of Marshallian economy, we follow Novshek and Sonnenschein [11]. However, since the increasing returns are not focus of their paper, they do not include it in the characterization. Form our point of view, the increasing returns are one of the essential issue of the character of Marshallian economy.

will extend the welfare theorem of Chipman [3] explained above to the case of several consumers.

The paper is organized as follows. In Section 2, the model is presented and the definition of the competitive equilibrium is given. In an example, it is pointed out that if every firm has the same production set, the model is formally reduced to that of Chipman [3]. In section 3, existence of the equilibrium will be proved. In so doing, we use the standard technique of Shafer and Sonnenscein [16]'s existence theorem with externalities. In section 4, we will extend the Chipman's first welfare theorem to the case of several consumers. Heuristically, the point of the problem can be explained as follows. Imagine an Edgeworth box. Here we are in a situation that the equilibrium is not on the contract curve (the set of Pareto optimal allocations) because of the externalities. Thus the question is: Which point of the contract curve should be compared to the equilibrium? Defining the social welfare function as a weighted sum of the individual utility functions, the weights are one to one correspondence to the point on the contract curve. Therefore the question is equivalent to: Which weight should be used to compare the equilibrium with the optimum? Note that this problem did not arise in Chipman [3], since he assumed that there was only one consumer, so that the social welfare function coincided with the individual utility function. Fortunately, the fixed point mapping of Negishi [12] tells us the correct weight, and using this mapping, we can calculate the optimal allocation corresponding to the equilibrium. The final section concludes.

2 The Model

There are three categories of commodities, the final output (consumption goods $x \in \mathcal{R}^n$, the intermediate good $y \in \mathcal{R}$ and the primary foctor $z \in \mathcal{R}$. The intemediate good is produced from the primary factor and it is used to produce the consumption goods. The set of firms producing the intermediate good is assumed to be $\mathcal{R}_+ = [0, +\infty)$, the set of non-negative real numbers. The firm $\beta \in \mathcal{R}_+$ is characterized by the production set $\hat{Y}(\beta) = Y(\beta) \cup \{0\} (\subset \mathcal{R}^2)$, where $Y(\beta)$ is a closed subset of $\mathcal{R}_+ \times \mathcal{R}_-$ and its generic element is denoted by $\eta(\beta) = (y(\beta), -z(\beta)) \in \mathcal{R}_+ \times \mathcal{R}_-$. See **Fig. 1**.

The nonconvexity of $\hat{Y}(\beta)$ allows "set-up cost" and it yields the "U-shaped" average cost curve. Consequently, free entry and exit of firms are represented as follows. Given price $\hat{q} = (q, w) \in \mathcal{R}^2_+$, where q is the intermediate good price and w is the wage rate, the firm β maximizes the profit $\pi = \hat{q}\eta$ in $\hat{Y}(\beta)$. If $\pi > 0$ for some $\eta \in \hat{Y}(\beta)$, the firm β takes a production activity in $Y(\beta)$; if $\pi < 0$ for all $\eta \in Y(\beta)$, it chooses $(0,0) \in \hat{Y}(\beta)$, which means the firm leaves the market.

For every $j = 1, \ldots, n$, the final output x^j is produced by the industry j which has the production function

$$f_j : \mathcal{R}_+ \times \mathcal{R}_+, (y, \gamma) \mapsto f_j(y, \gamma),\ j = 1, \ldots, n. \tag{1}$$

The final output industry takes γ as given when it maximizes the profit, but actually γ is

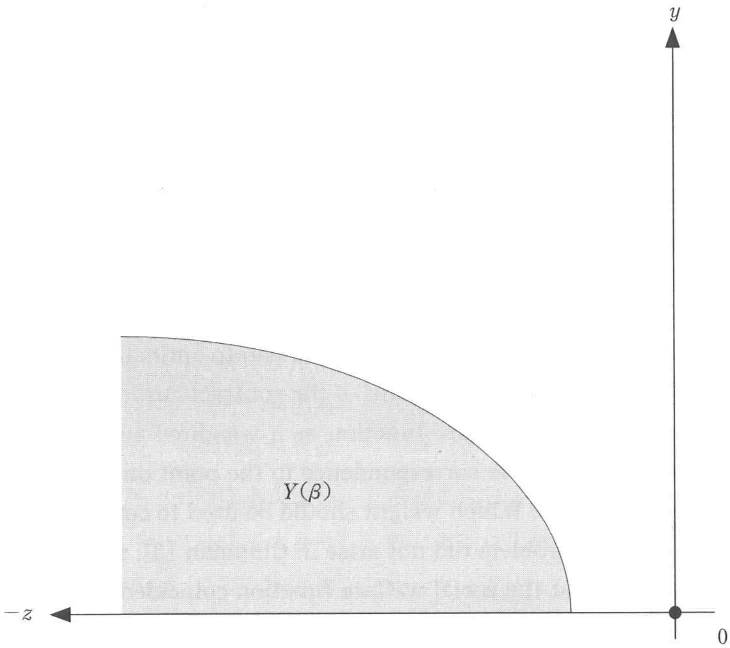

Fig. 1

determined endogenously at the level of (aggregate) input level, so that $y = \gamma$ must hold in equilibrium. We assume that the function f_j satisfies the following properties.

The function f_j is of degree of homogeneous one in y:

$$f_j(\lambda y, \gamma) = \lambda f_j(y, \gamma) \text{ for every } \lambda \geq 0, \tag{2}$$

and it is increasing in in γ:

$$f_j(y, \gamma) \geq f_j(y, \gamma') \text{ whenever } \gamma \geq \gamma'. \tag{3}$$

Then by these assumptions, the final output industry operates under the constant returns subjectively, but objectively it operates under the "increasing returns", since

$$f_j(\lambda y, \lambda\gamma) = \lambda f_j(y, \lambda\gamma) \geq \lambda f_j(y, \gamma) \text{ for every } \lambda \geq 1. \tag{4}$$

The consumption sector of the economy is standard; there exist m consumers and the consumer $i(=1, \ldots, m)$ is characterized by the consumption set $X_i \subset \mathcal{R}^{n+2}$ (recall that the commodity $n+1$ is the intermediate good, and the commodity $n+2$ is the primary factor), the preference $\geq_i$ which is complete and transitive binary relation on X_i, the endowment vector $\omega_i \in \mathcal{R}_+^{n+2}$, and the share of the firm β, $\theta_i(\beta) \geq 0$ with $\sum_{i=1}^{n} \theta_i(\beta) = 1$ a.e. As usual we set $<_i = X_i \times X_i \setminus \geq_i$.

The list $(\geq_i, \omega_i, \theta_i, f_j, Y)$ is called a *Marshallian economy* and denoted by ε_M. An allocation of the economy ε_M is an m+n+1-tuple $((\xi_i), (y_j), \eta)$ consisting of consumption vectors $\xi_i \in \mathcal{R}_+^{n+1}$, $i = 1, \ldots, m$, input of the industry j, $y_j(\geq 0)$, and an integrable function η: $\mathcal{R}_+ \rightarrow$

$\mathcal{R}_+ \times \mathcal{R}_-$, $\eta(\beta) = (y(\beta), -z(\beta))$. Given $(\gamma_i) = (\gamma_1, \ldots, \gamma_m) \in \mathcal{R}^n_+$, an allocation is said to be feasible for (γ_j) if $\sum_i \xi_i = \zeta((\gamma_j)) + \sum_i \omega_i$, where $\zeta((\gamma_i)) = (f_1(y_1, \gamma_1), \ldots, f_n(y_n, \gamma_n))$, $\int y(\beta) d\beta - \sum_{j=1}^{n} y_j$, $-\int z(\beta) d\beta)$.

A triple $\pi = (p, q, w) \in \mathcal{R}^{n+2}_+$ of the final output price p, the intermediate good price q, and the wage rate w is called a price system. For the price system (p, q, w), we sometimes write $\hat{p} = (p, w)$, $\hat{q} = (q, w)$.

Now we stste the definition of equilibrium.

Definition : An allocation $((\xi_i), (y_j), \eta)$ and a price system $\pi = (p, q, w)$ are said to be a *competitive equilibrium* of the economy ε_M if

(E-1) $\pi\xi_i \leq \pi\omega_i + \int \theta_i(\beta) \hat{q}\eta(\beta) d\beta$, and $\xi_i \geq_i \xi$ whenever $\pi\xi \leq \pi\omega_i + \int \theta(\beta) \hat{q}\eta(\beta) d\beta$, $i = 1, \ldots, m$

(E-2) $\hat{q}\eta \leq \hat{q}\eta(\beta)$ for every $\eta \in \hat{Y}(\beta)$, a.e.,

(E-3) $p^j f_j(y, y_j) - qy \leq p^j f_j(y_j, y_j) - qy_j = 0$ for every $y \geq 0, j = 1, \ldots, n$,

(E-4) $\sum_i \xi = \zeta((y_i)) + \sum_i \omega_i$.

The economic meaning is clear enough. The condition (E-1) is the standard utility maximizing condition, (E-2) and (E-3) are profit conditions for the final output and the intermediate good industries, respectively. Finally (E-4) means that the supply is equal to demand in all markets.

Next we give two simple examples and compute equilibria for the case $n = 1$.

Example 1 : Let $\hat{Y}(\beta) = \{(1, -\beta)\} \cup \{0, 0\}$. The firm β's decision is just to (a) produce 1 amount of the intermediate good from β amount of the labor or (b) choose (0, 0), i.e., leave the market. Firms are distributed uniformly from $\beta = 0$ with the highest productivity (it can produce from nothing!) to the lower productive firm. See **Fig. 2.**

The production function of the final output industry is given by $f(y, \gamma) = y\gamma$, which is of the increasing returns to scale explained above. There exists one consumer who has e amount of the primary factor and the utility function $u(x, z) = x^\alpha$. Hence he/she supplies the factor inelastically.

For (q, w) given, $\pi(\beta) = q - w\beta \geq 0$ for $0 \leq \beta \leq q/w$. Therefore the total net supply is

$$\int_0^{q/w} (1, -\beta) d\beta = (q/w, -(1/2)(q/w)^2). \tag{5}$$

Then by the resource constraint condition, we have $(1/2)(q/w)^2 = e$, hence $q/w = \sqrt{2e}$, From this, one obtains the consumption $x = f(\int y(\beta) d\beta, \int y(\beta) d\beta) = (\int y(\beta) d\beta)^2 = (q/w)^2 = 2e$. From the profit condition $pf(1, \int y(\beta) d\beta) = q$, it follows that $p/w = q/wf(1, \int y(\beta) d\beta) = 1$.

Example 2 : (Chipman [3]): This example is simpler than the previous one, but it is important in Section 4. Suppose that all firms β have that same production set $\hat{Y}(\beta) = \{(1, -1)\} \cup \{(0, 0)\}$. Then the total production set is

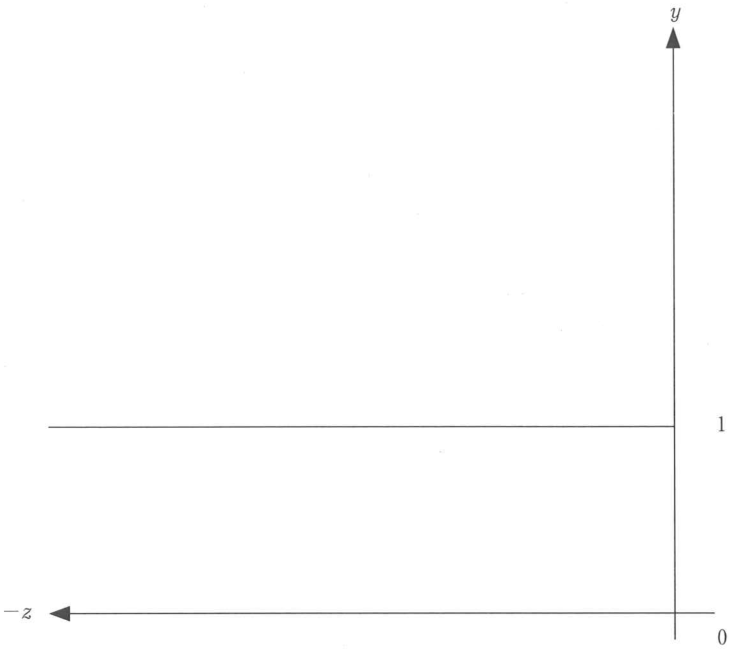

Fig. 2

$$\hat{Y} = \int \hat{Y}(\beta)d\beta = \left\{\int g(\beta)d\beta \,|\, g(\beta) \in \hat{Y}(\beta) a.e.\right\}$$
$$= \{\lambda(B)(1, -1) \,|\, B \in \mathcal{B}\} = \{t(1, -1) \,|\, t \geq 0\}, \quad (6)$$

where B is the set with $g(\beta) = (1, -1)$ for every $\beta \in B$, and $\mathcal{B}$ is the set of all Borel subsets of $\mathcal{R}_+$. (6) shows that the total production set is constant return to scale, hence equilibrium relative price should be $q/w = 1$, see **Fig. 3.**

Suppose that the final output industry and the consumption sector are the same as Example 1; the production set is given by $f(y, \gamma) = y\gamma$, and there exists one consumer with e amount of the labor. Then the net supply of the intermediate good industry is $(e, -e)$ and the consumption level is determined by $x = f(e, e) = e^2$, and the equilibrium (relative) price of the final output is $p/w = q/wf(1, e) = 1/e$.

Note that this model looks like an economy producing the final output directly from the primary factor. The formal structure of it is the same as that of Chipman [3]. However, the interpretation is different. In Chipman's model, the primary factor is used by (fixed) finitely many firms and the total quantity is the sum over all firms each of which may use different level of quantity. On the other hand, in our model, each firm uses one unit of the factor (and produces one unit of intermediate good), and the total amount is determined by the number (mass) of the firms which are active.

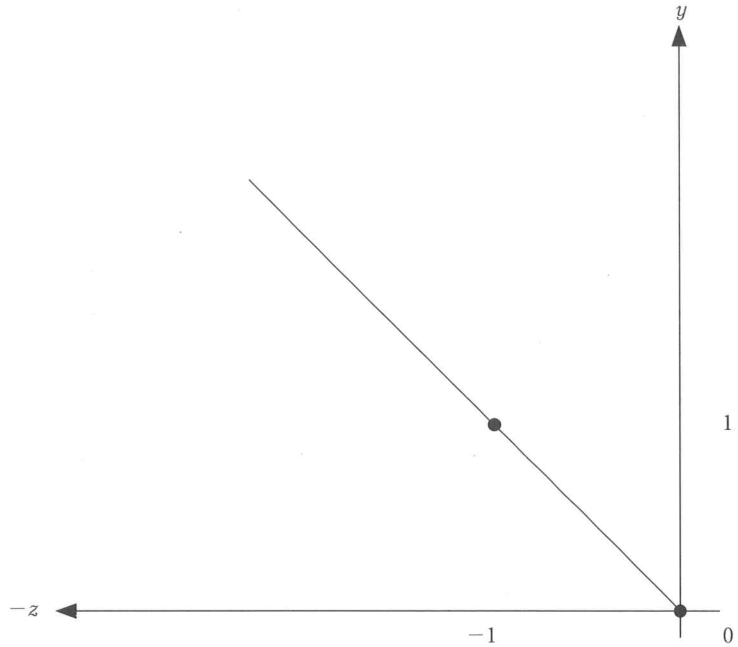

Fig. 3

3 Existence of Equilibrium

Let $\hat{Y} = \int \hat{Y}(\beta)d\beta$ be the total production set. By definition, the integral of the correspondence $\hat{Y}(\beta)$ is $\int \hat{Y}(\beta)d\beta = \{\int y(\beta)d\beta \in \mathcal{R}^2 | y: \mathcal{R}_+ \to \mathcal{R}^2, y(\beta) \in \hat{Y}(\beta) a.e.\}$.[4] Then by Richter's theorem (Hildenbrand [7, p. 62]), it is convex. Note that $0 \in \hat{Y}$, since $0 \in \hat{Y}(\beta) a.e.$ Our existence theorem reads as follows.

Theorem 1 : Suppose that the following conditions hold.

For every $i=1,\ldots,m$,

(C-1) X_i is closed, convex and bounded below,

(C-2) the set $\{(x,z) \in X_i \times X_i | x \succsim_i z\}$ is closed in $X_i \times X_i$, and for every $x \in X_i$, the set $\{z \in X_i | z \succsim_i x\}$ is convex.

(C-3) there is no $x \in X_i$ such that $x \succsim_i z$ for all $z \in X_i$

(C-4) $w_i \in$ interior X_i,

(C-5) the function $\theta_i: \mathcal{R}_+ \to \mathcal{R}_+$ is measurable,

4 For the properties of the integral of the correspondence, see Hildenbrand [7].

(F) the function $f_i: \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}_+$ is continuous, j=1,..., n,
(Y-1) the correspondence $\beta \to \hat{Y}(\beta)$ has a measurable graph,
(Y-2) the set $\hat{Y} = \int \hat{Y}(\beta) d\beta$ is nonempty and closed,
(Y-3) $\hat{Y} \cap \mathcal{R}_+^2 = \{0\}$

Then there exists a competitive equilibrium for ε_M.

Proof : In order to prove Theorem 1, we consider the variable profit assignment Arrow-Debreu economy (Novshek and Sonnenschein [10]). Given an economy $(X_i, \succeq_i, \omega_i, \theta_i, f_j, Y)$, the associated variable profit assignments economy $\varepsilon_{AD} = (X_i, \succeq_i, \omega_i, w_i, f_j, \hat{Y})$ is defined by

$$\hat{Y} = \int \hat{Y}(\beta) d\beta, \tag{7}$$

and

$$w_i(\hat{q}) = \int \theta_i(\beta) \sup\{\hat{q}\eta \,|\, \eta \in \hat{Y}(\beta)\} d\beta, \tag{8}$$

where $\hat{q} = (q, w)$.

An equilibrium for the variable profit assignments A-D economy $\varepsilon_{AD} = (X_i, \succeq_i, \omega_i, w_i, f_j, \hat{Y})$ is a pair consisting of an allocation $((\xi_i), (y_j), \eta = (y, -z))$ and a price system $\pi = (p, q, w)$ satisfying:

(E-1') $\pi\xi_i \leq \pi\omega_i + w_i(\hat{q})$, and $\xi_i \succeq_i \xi$ whenever $\pi\xi \leq \pi\omega_i + w_i(\hat{q})$, $i = 1, \ldots, m$,
(E-2') $\hat{q}\eta' \leq \hat{q}\eta$ for ever $\eta' \in \hat{Y}$,
(E-3') $p^j f_j(y, y_j) - qy \leq p^j f_j(y_j, y_j) - qy_j = 0$ for every $y \geq 0, j = 1, \ldots, n$,
(E-4') $\sum_i \xi_i = \zeta((y_j)) + \sum_i \omega_i$,
where $\zeta((y_j)) = (f_1(y_1, y_1), \ldots, f_n(y_n, y_n), y - \sum_j y_j, -z)$.

Now we can prove

Lemma 1 : Under the assumptions of Theoreml, there exists an equilibrium for the associated variable profit assignments economy ε_{AD}.

Proof : Appendix.

And we need

Lemma 2 (Novshek and Sonnenschein [10]) : If the graph of the correspondence $\beta \mapsto \hat{Y}(\beta)$ is measurable and $\hat{Y} = \int \hat{Y}(\beta) d\beta \neq \emptyset$, then for ever $p \in \mathcal{R}^{n+2}$, $\sup\{p\eta \,|\, \eta \in \int \hat{Y}(\beta) d\beta\} = \int \sup\{p\eta \,|\, \eta \in \hat{Y}(\beta)\} d\beta$.

Now let $((p, q, w), (x_i), (y_j), \eta)$ be an equilibrium for ε_{AD}. Then by the difinition, the conditions (E-1) and (E-3) are met. Since $\eta \in \int \hat{Y}(\beta) d\beta$, we have an integrable function

$\eta : \mathcal{R}_+ \to \mathcal{R}_+ \times \mathcal{R}_-$ such that $\eta(\beta) \in \hat{Y}(\beta)$ a.e. and $\eta = \int \eta(\beta) d\beta$. By Lemma 2, $\hat{q}\eta(\beta) = \sup\{\hat{q}\eta \mid \eta \in \hat{Y}(\beta)\}$ a.e, where $\hat{q} = (q, w)$. Hence (E-2) is met. Since $\eta = \int \eta(\beta) d\beta$, (E-4) follows from (E-4'). Therefore $((p, q, w), (x_i), (y_j), \eta)$ is an equilibrium for ε_M. ▮

4 Optimality of Equilibrium

In this section, we consider the Pareto optimality of equilibrium. Throughout the section, we assume that the model is that of Chipman [3]'s type, that is, every firm β has the production set

$$\hat{Y}(\beta) = \{(1, -1), (0, 0)\}, \tag{9}$$

then the total production set $\hat{Y} = \int \hat{Y}(\beta) d\beta$ is constant returns to scale,

$$\hat{Y} = \{(t, -t) \mid t \geq 0)\}. \tag{10}$$

Hence in equilibrium, it follows that $q = w$, and the economy behaves as if there are no intermediate goods and the final output is produced from the primary factor directly. See Example 2 of Section 2.

Set $f_j(y, y) = \phi_j(y)$. Then the elasticity of industry j is given by $\epsilon_j(y) = \phi_j'(y) y / \phi(y)$[5].

As in Chipman [3], we assumed that the utility function is of Cobb-Douglas and the consumer i has the utility function

$$u_i(x_i) = \sum_{j=1}^{n} \alpha_i^j \log x_i^j, \ \alpha_i^j \geq 0, \ \sum_{j=1}^{n} \alpha_i^j = 1 \tag{11}$$

and the initial endowment $\omega_i = (0, \dots, 0, e)$. Note that the utility function does not depend on x_i^{n+1} and x_i^{n+2}. This assumption combined with the form of initial endowment vector implies that every consumer supplies e amount of the primary factor inelastically.

Let the total resource be $Z = me$. We are interested in the condition under which the competitive equilibrium coincides with the Pareto optimal allocation. For this purpose, first we shall characterize the equilibrium as a fixed point of a mapping which was developed by Negishi [12].

Given $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathcal{R}_+^n$, consider the constrained optimization problem

$$P((\gamma_i)) : \text{Max} \sum_{i=1}^{m} \sigma_i \sum_{j=1}^{n} \alpha_i^j \log x_i^j \text{ subject to}$$

$$\sum_{i=1}^{n} x_i^j \leq f_j(y_j, \gamma_j), \ j = 1, \dots, n,$$

$$\sum_{j=1}^{n} y_j \leq Z, \tag{12}$$

where $\sigma_1, \dots, \sigma_m$ are weights of consumers satisfying $\sigma_i \geq 0$, $\sum_{i=1}^{n} \sigma_i = 1$. By the Kuhn-Tucker theorem, the solution of this problem is a saddle point of the Lagrangian

$$\mathcal{L}_{\sigma,\gamma}((x_i), (y_j), p, w) = \sum_{i,j} \alpha_i^j \log x_i^j + \sum_j p^j (f_j(y_j, \gamma_j) - \sum_i x_i^j) + w(Z - \sum_j y_j). \tag{13}$$

5 Chipman [3] assumed that $\phi_j(y) = \kappa_j y_j^{\rho_j}$, where κ_j and ρ_j and constants, so that the elasticity is constant and equal to ρ_j. He called ρ_j the degree of homogeneity.

Let the saddle point be $((x_i^*), (y_j^*), p^*, w^*)$. Since the set of feasible allocations is bounded, we can take a constant $b > 0$ such that

$$\max\left\{\sum_i |we - px_i| \,\middle|\, \sum_i x_i \le f_j(y_j, \gamma_j),\ j = 1, \ldots, n,\ \sum_j y_j \le Z\right\} < b. \tag{14}$$

Given $((\sigma_i), (x_i^j), (p^j), w)$, define (σ_i^*) by

$$\sigma_i^* = \frac{\max\{0, \sigma_i + (1/b)(we - px_i)\}}{\sum_i \max\{0, \sigma_i + (1/b)(we - px_i)\}},\ i = 1, \ldots, m. \tag{15}$$

Then the competitive equilibrium is the fixed point of the mapping Φ defined by[6]

$$\Phi : ((\sigma_i), (\gamma_j), (x_i), (y_j), p, w) \mapsto ((\sigma_i^*), (y_j^*), (x_i^*), (y_j^*), p^*, w^*). \tag{16}$$

Setting $(\sigma_i) = (\sigma_i^*)$ and $(\gamma_j) = (y_j^*)$ in the first order conditions for $\mathcal{L}_{\sigma,\gamma}$, we have

$$\sigma_i^* (\alpha_i^j / x_i^{*j}) - p^{*j} = 0,\ i = 1, \ldots, m,\ j = 1, \ldots, n, \tag{17}$$

$$p^{*j} f_j(y_j^*, y_j^*) - w^* y_j^* = 0,\ j = 1, \ldots, n. \tag{18}$$

The conditions (17) and (18) are nothing but the utility maximization and profit maximization conditions, respectively. Summing (17) over j and using $p^* x_i^* = w^* e$, we obtain

$$\sigma_i^* = w^* e,\ i = 1, \ldots, m. \tag{19}$$

This is a well known condition: in equilibrium, the weight of a consumer is the inverse of his marginal utility of income (Negishi [12, p. 97]).

Summing (17) over i with the help of (18) and (19), $w^* (\sum_i \alpha_i^j)\, e = p^{*j} \sum_i x_i^{*j} = p^{*j} \phi_j (y_j^*) = w^* y_j^*$, hence

$$y_j^* = \sum_i \alpha_i^j e,\ j = 1, \ldots, n, \tag{20}$$

and from this, one obtains

$$p^{*j} / w^* = y_j^* / \phi\,(y_j^*),\ j = 1, \ldots, n, \tag{21}$$

$$x_i^{*j} = (w^* / p^{*j}) \alpha_i^j e = (\alpha_i^j / \sum_i \alpha_i^j)\, \phi_j\,(y_j^{*j}),\ i = 1, \ldots, m,\ j = 1, \ldots, n. \tag{22}$$

Therefore the competitive equilibrium is given by (20), (21) and (22).

Now consider the social optimization problem

$$\begin{aligned} P:\ & \operatorname{Max} \sum_i \sigma_i \sum_j \alpha_i^j \log x_i^j \text{ subject to} \\ & \sum_i x_i^j \le \phi_j\,(y_j),\ j = 1, \ldots, n, \\ & \sum_j y_j \le Z. \end{aligned} \tag{23}$$

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6 In earlier stage of the research, the author carried out this idea in detail to prove the existence of equilibrium for the Chipman model before he applied the game theoretic approach of Shafer and Sonnenschein.

The problem does not depend on (γ_j) any more. On account of (19), we set $\sigma_1 = \ldots = \sigma_m = w^* e$ to compare the equilibrium and an optimum. In words, we compare two allocations with the same social welfare function. The first order conditions are

$$w^* e\,(\alpha_i^j / x_i^{oj}) - \lambda^j = 0,\ i = 1, \ldots, m,\ j = 1, \ldots, n, \tag{24}$$

$$\lambda^j \phi_j'\,(y_j^o) - \nu,\ j = 1, \ldots, n, \tag{25}$$

where λ^j and ν are multipliers. Summing (24) over i, we have

$$\lambda^j = w^* e \sum_i \alpha_i^j / \sum_i x_i^{oj} = w^* e \sum_i \alpha_i^j / \phi_j\,(y_j^o),\ j = 1, \ldots, n. \tag{26}$$

Substituting (26) into (25),

$$w^* e \sum_i \alpha_i^j\,(\phi_j'\,(y_j^o) / \phi_j\,(y_j^o)) = \nu,\ j = 1, \ldots, n. \tag{27}$$

Multiplying y_j^o with (27) and summing over j,

$$\nu = (w^* e / Z) \sum_i \sum_j \alpha_i^j\,(\phi_j'\,(y_j^o) y_j^o / \phi_j\,(y_j^o)). \tag{28}$$

Substituting (28) into (27), one obtains

$$y_j^o = \left(\frac{\epsilon_j\,(y_j^o)}{(1/m) \sum_i \sum_j \alpha_i^j \epsilon_j\,(y_j^o)} \right) \sum_i \alpha_i^j e. \tag{29}$$

From (24) and (26), the optimal consumptions are determined by

$$x_i^{oj} = \left(\frac{\alpha_i^j}{\sum_i \alpha_i^j} \right) \phi_j\,(y_j^o),\ i = 1, \ldots, m,\ j = 1, \ldots, n. \tag{30}$$

According to (20), (22), (29) and (30), we have

$$x_i^{oj} \leq x_i^{*j} \Leftrightarrow \phi_j\,(y_j^o) \leq \phi\,(y_j^*) \Leftrightarrow y_j^o \leq y_j^* \Leftrightarrow \epsilon_j\,(y_j^o) \leq (1/m) \sum_i \sum_j \alpha_i^j \epsilon_j\,(y_j^o). \tag{31}$$

Therefore we have obtained the next theorem which was first stated by Chipman [3, p. 365] for the case $m = 1$.

Theorem 2 : *Suppose that every consumer has the Cobb-Douglas utility function (11) and the same amount of labor as endowment. Then, optimal output of the j-th product is greater than, equal to, or less than competitive output*[7] *according as the elasticity of the industry (at the optimum) is greater than, equal to, or less than the weighted average of elasticities of all industries. In particular, if all industries' elasticities are equal to the weighted average, the competitive equilibrium is Pareto optimal.*

7 Laissez-faire output in Chipman's terminology.

5 Conclusions

1. For simplicity, we have assumed that there exist only one factor and one intermediate good. It is clear that there are no difficulties to extend to the case that there exist finitely many factors and intermediate goods. An interesting and challenging problem is that the commodities are perfectly differentiated, so that there are infinitely many (continuum of) goods (or factors). Romer [14] analyzed such a market in which the intermediate goods are subject to commodity differentiation. This formulation is important from the Marshallian point of view, since according to Marshall, one reason of external economies is the increase of productivity due to specialization such as the division of labor, and the commodity differentiation of the input goods naturally represents the specialization.

2. Related to this point, the study of other (not competitive or imperfectly competitive) equilibrium concepts such as the core, Cournot equilibrium, etc... should be of interest. For example, since the competitive equilibrium with external increasing returns is not generally Pareto optimal (see Theorem 2), the existence of competitive equilibrium (Theorem 1) does not imply the nonemptiness of the core which is a subset of Pareto optimal allocations by definition. Hence the existence of the core in this context is an open question.

3. With regard to the question of the compatibility between the increasing returns and competitive equilibrium, we would like to say that our solutions given by the present paper and [18] are definitive. A reason that this problem was so formidable in the 1920's and 30's is that there was a basic confusion between externalities and increasing returns; people often mixed up a nonconvex technology and increasing returns. However, the modern general equilibrium theory removed the confusions and provided technique to analyze the dynamic market structures such as the market with infinite time horizon of Young [19]'s type or that with firms' free entry and exit of Marshall [9]'s type. Furthermore, the work of Chipman and Romer facilitated our understanding of the increasing returns. We synthesized these accomplishments of the past and solved a few technical problems. Now we know that the theoretical foundations of the idea is solid and the picture of Marshall and Young which placed the increasing returns in a central role for the economic progress is correct. We hope our result will be a stepping stone for further investigations in the future.

Appendix

In this appendix, we will prove

Lemma 1: Suppose that an economy $\mathcal{E}_M = (X_i, \succeq_i, \omega_i, \theta_i, f_j, Y)$ satisfies the following conditions.

For every $i = 1, \dots, m$,

(C-1) X_i is closed, convex and bounded below,

(C-2) the set $\{(x, z) \mid x \succeq_i z\}$ is closed in $X_i \times X_i$, and for every $x \in X_i$, the set $\{z \in X_i \mid z \succeq_i x\}$ is convex,

(C-3) there is no $x \in X_i$ such that $x \succeq_i z$ for all $z \in X_i$,

(C-4) $\omega_i \in$ *interior* X_i,

(C-5) the function $\theta_i : \mathcal{R}_+ \to \mathcal{R}_+$ is measurable,

(F) the function $f_j : \mathcal{R}_+ \times \mathcal{R}+$ is continuous, $j = 1, \ldots, n$,

(Y-1) the correspondence $\beta \to \hat{Y}(\beta)$ has a measurable graph,

(Y-2) the set $\hat{Y} = \int \hat{Y}(\beta) d\beta$ is nonempty and closed,

(Y-3) $\hat{Y} \cap \mathcal{R}^2_+ = \{0\}$.

Then there exists a competitive equilibrium for the associated variable profit assignments A-D economy $\mathcal{E}_{AD}$.

Proof : In order to prove Lemma 1, first we will show that

Claim 1 : $\sup\{y \geq 0 \mid (y, z) \in \hat{Y}, 0 \leq z \leq \Sigma_i \omega_i^{n+2}\} < +\infty$

Proof : Suppose not. Then there exists a sequence such that $\eta_n = (y_n, z_n) \in \hat{Y}, 0 \leq z_n \leq \sum_i \omega_i^{n+2}$, and $y_n \to +\infty$. Let $\hat{\eta}_n = \eta / |y_n|$. Since $\hat{Y}$ is convex and $0 \in \hat{Y}$, $\hat{\eta}_n \in \hat{Y}$. It can be easily seen that $\hat{\eta}_n \to (1, 0)$. Since $\hat{Y}$ is closed by the assumption (Y-2), we have $(1, 0) \in \hat{Y}$. This contradicts the assumption (Y-3). ∎

Next we want to show that

Claim 2 : The set of feasible allocations

$$A = \left\{((\xi_i), (y_j), (y, -z)) \in \prod_i X_i \times \mathcal{R}^n_+ \times \hat{Y} \middle| \sum_i \xi_i = \zeta((y_j)) + \sum_i \omega_i\right\},$$

where $\zeta((y_j)) = (f_1(y_1, y_1), \ldots, f_n(y_n, y_n), y - \sum_j y_j, -z)$, is bounded.

Proof : Let $((\xi_i), (y_j), (y, -z)) \in A$. Then it is clear that $0 \leq z \leq \sum_i \omega_i^{n+2}$. By Claim 1, we have $0 \leq y \leq \bar{y} \equiv \sup\{y \geq 0 \mid (y, z) \in \hat{Y}, 0 \leq z \leq \sum_i \omega_i^{n+2}\}$. Thus for every j, $0 \leq y_j \leq \bar{y} + \sum_i \omega_i^{n+1}\}$, and for every i, $b_i \leq \xi_i \leq \sum_{j \neq i}(-b_j) + \zeta((\bar{y})) + \sum_i \omega_i$, where b_i is a constant with $b_i \leq \xi_i$ for every $\xi_i \in X_i$, $i = 1, \ldots, m$. ∎

By Claim 2, we can take a compact interval $J \subset \mathcal{R}$ such that $A \subset$ *interior* K, where $K \equiv J^{(n+2)m+n+2}$.

We need

Lemma 3 : (Shafer and Sonnenschein [15]) : Let an abstract game $\Gamma = (X_i, P_i, \mathcal{A}_i)$ satisfy the following:

(α) Each X_i is a nonempty, compact and convex subset of $\mathcal{R}^k$.

(β) Each P_i is a preference crrespondence $P_i : \prod_i X_i \to X_i$ such that

(β-1) P_i has an open graph in $\prod_i X_i \times X_i$.

(β-2) for every $x = (x_i) \in \prod_i X_i$, $x_i \notin conv\ P_i(x)$, and[8]

(γ) each $\mathcal{A}_i$ is a constraint correspondence $\mathcal{A}_i : \prod_i \to X_i$ such that

(γ-1) $\mathcal{A}_i$ is a continuous correspondence,

(γ-2) for every $x \in \Pi_i X_i$, $\mathcal{A}_i(x)$ is nonempty, compact and convex.

Then there exists an equilibrium for Γ, i.e., there exists an $\bar{x} = (\bar{x})$ such that for each i,

(δ-1) $\bar{x} \in \mathcal{A}_i(\bar{x})$,

(δ-2) $P_i(\bar{x}) \cap \mathcal{A}_i(\bar{x}) = \emptyset$.

The remainder of the proof will proceed along a familiar path (e.g., Arrow and Debreu [1] or Shafer and Sonnenschein [16]). We convert $\mathcal{E}_{AD}$ into an $m+n+2$ person game $\hat{\Gamma}$ with the strategy space $\mathcal{R}^{n+2}$. For $i = 1, \ldots, m$, we set $\hat{X}_i = X_i \cap J^{n+2}$. Netx for $j = 1, \ldots, n$, define

$$\hat{X}_{m+j} = \left\{ (x^k) \in \mathcal{R}^{n+2} \middle| x^k = 0 \text{ for all } k \text{ but } k = m+j \text{ or } n+1,\ x^{m+j} \geq 0,\ x^{n+1} \leq 0 \right\}, \tag{32}$$

and

$$\hat{X}_{m+n+1} = \left\{ (x^k) \in \mathcal{R}^{n+2} \middle| x^k = 0 \text{ for all } k \text{ but } k = n+1 \text{ or } n+2,\ x^{n+1} \geq 0,\ x^{n+2} \leq 0 \right\}. \tag{33}$$

Finally, we define

$$\hat{X}_{m+n+2} = \left\{ \pi = (p^k) \in \mathcal{R}^{n+2}_+ \middle| \sum_{k=1}^{n+2} p^k = 1 \right\}. \tag{34}$$

Sometimes we write for $(p^k) \in \hat{X}_{m+n+2}$, $p^{n+1} = q$, $p^{n+2} = w$. Obviously, $\hat{X}_i$ is compact and convex subset of $\mathcal{R}^{n+2}$, $i = 1, \ldots, m+n+2$.

The first m players are described as follows. The player i has the choice set $\hat{X}_i$, the constraint correspondence $\hat{\mathcal{A}}_i : \Pi_i \hat{X}_i \to \hat{X}_i$ defined by

$$\hat{\mathcal{A}}_i((x_i), (y_j), \pi) = \left\{ z \in \hat{X}_i \middle| \pi z \leq \pi \omega_i + \pi y_{m+n+1} \right\}, \tag{35}$$

and the preference correspondence $\hat{P}_i : \prod_i \hat{X}_i \to \hat{X}_i$ defined by

$$\hat{P}_i((x_i), (y_j), \pi) = \left\{ z \in \hat{X}_i \middle| x_i \prec_i z \right\}, \tag{36}$$

By (C-4), the correspondence $\mathcal{A}_i$ is continuous (Debreu [4, p. 63]), and clearly it is nonempty, compact and convex valued. By (C-2), $\hat{P}_i$ has an open graph and $x_i \notin conv\ \hat{P}_i((x_i), (y_j), \pi)$. The players $j = 1, \ldots, n$ are described as follows. The constraint correspondence $\hat{\mathcal{A}}_j : \Pi_i \hat{X}_i \to \hat{X}_{m+j}$ is

$$\hat{\mathcal{A}}_j((x_i), (y_j), \pi) = \left\{ z = (z^k) \in \hat{X}_{m+j} \middle| z^{m+j} \leq f_j(z^{n+1}, -y_j^{n+1}) \right\}, \tag{37}$$

and the preference correspondence $\hat{P}_i : \prod_i \hat{X}_i \to \hat{X}_{m+j}$ is given by

$$\hat{P}_j((x_i), (y_j), \pi) = \left\{ z \in \hat{X}_{m+j} \middle| \pi z > \pi y_j \right\}. \tag{38}$$

By (F), it is easy to show that the correspondence $\hat{\mathcal{A}}_j$ is continuous. Obviously, it is nonemp-

8 $conv\ A$ means the convex hull of the set A.

ty and compact valued. It is clear that $\hat{P}_j$ has an open graph and $y_j \notin conv\ \hat{P}_j((x_i), (y_j), \pi)$. The player $m+n+1$ has the constraint correspondence $\hat{\mathcal{A}}_{m+n+1}: \prod_i \hat{X}_i \to \hat{X}_{m+n+1}$ defined by

$$\hat{\mathcal{A}}_{m+n+1}((x_i), (y_j), \pi) = \left\{z = (z^k) \in \hat{X}_{m+n+1} \middle| (z^{n+1}, z^{n+2}) \in \hat{Y}\right\}. \tag{39}$$

Clearly it is nonempty, compact and convex valued, continuous correspondence. Its preference correspondence is the same as that of the player j and it is given by

$$\hat{P}_{m+n+1}((x_i), (y_j), \pi) = \left\{z \in \hat{X}_{m+n+1} \middle| \pi z > \pi y_{m+n+1}\right\}. \tag{40}$$

As stated above, it has an open graph and $y_{m+n+1} \notin conv\ \hat{P}_{m+n+1}$. The last player, called "the market player" has the constraint correspondence

$$\hat{\mathcal{A}}_{m+n+2}((x_i), (y_j), \pi) = \hat{X}_{m+n+2}. \tag{41}$$

Since the $\hat{\mathcal{A}}_{m+n+2}$ is constant, it is continuous. Obviously it is nonempty, compact and convex valued. Its preference correspondence $\hat{P}_{m+n+2}: \prod_i \hat{X}_i \to \hat{X}_{m+n+2}$ is given by

$$\hat{P}_{m+n+2}((x_i), (y_j), \pi) = \left\{\bar{\pi} \in \hat{X}_{m+n+2} \middle| \bar{\pi}\left(\sum_i x_i - \zeta - \sum_i \omega_i\right) > \right.$$

$$\left. \pi\left(\sum_i x_i - \zeta - \sum_i \omega_i\right)\right\}, \text{ where } \zeta = \left(y_1^1, \ldots, y_n^n, y_{n+1}^{n+1} - \sum_j y_j^{n+1}, y_{n+1}^{n+2}\right). \tag{42}$$

It is easy to verify that $\hat{P}_{m+n+2}$ has an open graph and $\pi \notin conv\ \hat{P}_{m+n+2}$. By Lemma 5, there exists an equilibrium $((\hat{x}_i), (\hat{y}_j), \hat{\pi})$ for $\hat{\Gamma}$. We can easily check that (E-1'), (E-2') and (E-3') are met. Then by summing the budget constraints over $i = 1, \ldots, m$, we have the Walras law; $\hat{\pi}(\sum_i \hat{x}_i - \hat{\zeta} - \sum_i \omega_i) \leq 0$, where $\hat{\zeta} = (\hat{y}_1^1, \ldots, \hat{y}_n^n, \hat{y}_{n+1}^{n+1} - \sum_j \hat{y}_j^{n+1}, \hat{y}_{n+1}^{n+2})$. Since the market player maximizes the value of the demand, it follows that $\pi(\sum_i \hat{x}_i - \hat{\zeta} - \sum_i \omega_i) \leq 0$ for every $\hat{\pi} \in \hat{X}_{m+n+2}$. Taking $\pi = (\delta_j^i)$, where $\delta_j^i = 1$ when $i = j$, and $= 0$ otherwise, we have $\sum_i \hat{x}_i - \hat{\zeta} - \sum_i \omega_i \leq 0$. Therefore (E-4') is met and the proof is established. ▮

References

1. K. J. Arrow and G. Debreu, Existence of an equilibrium for a competitive economy, *Econometrica* **22** (1954), 265–290.
2. J. S. Chipman, A survey of the theory of international trade: part I, the neoclassical theory, *Econometrica* **33** (1965), 685–760.
3. J. S. Chipman, External economies of scale and competitive equilibrium, *Quart. J. Econ.* **84** (1970), 347–385.
4. G. Debreu, "Theory of Value," Wiley, New York, 1959.
5. F. Graham, Some fallacies in the interpretation of social cost. A reply, *Quart. J. Econ.* **39** (1925), 324–330.
6. E. Helpman, Increasing returns, imperfect markets, and trade theory *in* "Handbook of International Economics," Vol. 2., North Holland, Amsterdam, 1984.
7. W. Hildenbrand, "Core and Equilibria of a Large Economy," Princeton Univ. Press, New Jersey, 1974.
8. F. H. Knight, Some fallacies in the interpretation of social cost, *Quart. J. Econ* **38** (1924), 582–606.
9. A. Marshall, "Principles of Economics," Macmillan, London, 1890.
10. W. Novshek and H. Sonnenschein, Small efficient scale as a foundation for Walrasian equilibrium, *J. Econ. Theory* **22** (1980), 243–255.
11. W. Novshek and H. Sonnenschein, General equilibrium with free entry: a synthetic approach to the

theory of perfect competition, *J. Econ. Literature* **25** (1987), 1291–1306.

12. T. Negishi, Welfare economics and existence of equilibrium for a competitive economy, *Metroeconomica* **23** (1960), 92–97.
13. P. Romer, Increasing returns and long-run economic growth, *J. Polit. Econ.* **94** (1986), 1002–1037.
14. P. Romer, Growth based on increasing returns due to specialization, *Am. J. Econ.* **77** (1987), 56–62.
15. W. Shafer and H. Sonnenschein, Equilibrium in abstract economies without ordered preferences, *J. Math. Econ.* **2** (1975), 345–348.
16. W. Shafer and H. Sonnenschein, Equilibrium with externalities, commodity taxation, and lump sum transfers, *Int. Econ. Rev.* **17** (1976), 601–611.
17. P. Sraffa, The laws of return under competitive conditions, *Econ. J.* **36** (1926), 535–550.
18. T. Suzuki, Intertemporal general equilibrium model with external increasing returns, *J. Econ. Theory* **69** (1996), 117–133.
19. A. Young, Increasing returns and economic progress, *Econ. J.* **38** (1928), 527–542.

（2004 年 5 月 7 日経済学会受理）