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On the Regularity of Equilibria for Exchange Economies: Finite Dimensional Case

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Ⅰ. Introduction

It is well known that in the theory of general market equilibrium, the global uniqueness of the equilibrium, although it is a nice and desirable property, can be obtained only with the cost of very strong assumptions (see, for instance, Arrow and Hahn [1971]). In a path breaking paper, G.Debreu [1970] started the study of the local uniqueness (finiteness) of competitive equilibria of exchange economies. He considered a topological space of all (smooth) economies and concluded that the economies with locally finite and structurally stable equilibria are open and dense subset of the space. Such economies are now called the regular economies. This result has been extended to several directions, for example, to production economies, the exchange economies with a continuum of traders and so on by Smale [1974], Dieker [1975], Mas-Colell [1975, 1977], Kehoe [1983], Rajan [1977] and many others. The reader can consult the book of Mas-Colell [1985] for related works and literatures.

On the other hand, Kehoe and Levine [1985], Balasko [1997] and Chichilnisky and Zhou [1998] and others have extended Debreu's result to economies with infinite dimensional commodity spaces.

The purpose of this note is to settle the methodological foundations in order to pursue the stream of these researches. In a subsequent paper, we will generalize the result of the present paper to the case of infinite dimensional commodity spaces. Our method is basically based on that of Balasko. He parameterized "the equilibrium manifold" by weights of a social welfare function (the weighted sum of individual utility functions) on Pareto optimal allocations, rather than (as usual) by the price vectors through the excess demand equation. This clever trick will make us to avoid the fundamental difficulty arising in the infinite dimensional settings mentioned above. This approach was started by Negishi [1960] for proving the existence of competitive equilibria and used by Kehoe and Levine [1985] for the study of the local uniqueness of dynamic equilibria.

The next section presents the model and results. In appendix, we collect the mathematical concepts and related results used in our analysis.

Ⅱ. The model and results

Consider an exchange economy with m consumers. The commodity space of our economy is l-dimensional Euclidean space $\mathbb{R}^l$.

The consumer i's consumption set X_i is defined by

$$X_i=\{\boldsymbol{x}=(x)^t\in\mathbb{R}^l|x^t>0\ \ t=1\ldots l\}\equiv\mathbb{R}^l_{++},\ i=1\ldots m. \tag{1}$$

As usual, The consumer i's preference is represented by a utility function

$$u_i:X_i\to\mathbb{R},\ \boldsymbol{x}\mapsto u_i(\boldsymbol{x}),\ i=1\ldots m. \tag{2}$$

The consumer i's initial endowment vector is denoted by $\omega_i=(\omega_i^1\ldots\omega_i^l)\in X_i$. The initial endowments profile is denoted by $\omega\equiv(\omega_1\ldots\omega_m)\in\prod_{i=1}^m X_i$. We will denote $\Omega=\prod_{i=1}^m X_i$ and as will be seen, we identify each vector $(\omega_1\ldots\omega_m)\in\Omega$ as an economy.

We assume that the utility function satisfies the following conditions.

(U-1) u_i is twice continuously differentiable, namely that of class C^2 on X_i.

Let $Du_i(\boldsymbol{x})=(\partial_0 u_i(\boldsymbol{x}),\partial_1 u_i(\boldsymbol{x}),\ldots\partial_l u_i(\boldsymbol{x}))$ be the derivative (tangent map) of u_i at $\boldsymbol{x}\in X_i$, and

$$D^2u_i(\boldsymbol{x})=\begin{bmatrix}\partial_0\partial_0u_i(\boldsymbol{x}) & \partial_0\partial_1u_i(\boldsymbol{x}) & \partial_0\partial_lu_i(\boldsymbol{x})\\ \partial_1\partial_0u_i(\boldsymbol{x}) & \partial_1\partial_1u_i(\boldsymbol{x}) & \partial_1\partial_lu_i(\boldsymbol{x})\\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots\\ \partial_l\partial_0u_i(\boldsymbol{x}) & \partial_l\partial_1u_i(\boldsymbol{x}) & \partial_l\partial_lu_i(\boldsymbol{x})\end{bmatrix} \tag{3}$$

be the second derivative, where $\partial_t u_i(\boldsymbol{x})=lim_{h\to0}(1/h)(u_i(x^0,\ldots,x^t+h,\ldots x^l)-u_i(x^0,\ldots,x^t,\ldots x^l))$, and so on. For every $\boldsymbol{x}\in X_i$, $D^2u_i(\boldsymbol{x})$ is considered to be a linear map from $\mathbb{R}^l$ to itself. We assume for every i,

(U-2) u_i is strictly differentiably monotone, i.e., $Du_i(\boldsymbol{x})\gg0$ for every $\boldsymbol{x}\in X_i$.

(U-3) $D^2u_i(\boldsymbol{x})$ is a nondegenerate, negative definite bilinear form on $\mathbb{R}^l$, namely that

$(\boldsymbol{y},D^2u_i(\boldsymbol{x})\boldsymbol{y})\leq0$ for every $\boldsymbol{y}\in\mathbb{R}^l$ and the equality holds only when $\boldsymbol{y}=0$.

(U-4) for every sequence $\boldsymbol{x}_n=(x_n^t)\in X_i$ such that $x_n^t\to0$ for some t, it follows that $u_i(x_n)\to-\infty$.

Note that the assumption (U-3) implies that the linear map $D^2u_i(\boldsymbol{x})$ is a linear isomorphism of $\mathbb{R}^l$ to itself. The assumption (U-4) will be sometimes called Inada condition.

The list $\{u_i,\omega_i\}_{i=1}^m$ is called an economy and denoted by E. From now on, we fix the utility functions and parameterize economy by the endowment vectors. Therefore the set $\Omega=\prod_i X_i$ is the space of all economies. Therefore we will often call an endowment profile $\omega=(\omega_i)$ simply an economy.

A price vector in the economies is an l-vector $\boldsymbol{p}=(p^t)\gg 0$. For a given commodity vector $\boldsymbol{x}=(x^t)$, the value of $\boldsymbol{x}$ evaluated by the price $\boldsymbol{p}$ is defined by the inner product $\boldsymbol{px}=\sum_{t=0}^{l} p^t x^t$. An m-tuple of consumption vectors $(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m)\in\prod_i X_i$ is called an allocation. The allocation is said to be feasible if $\sum_i x_i=\sum_i \omega_i$.

Definition 1: An allocation $(\boldsymbol{x}_i)$ is said to be Pareto optimal if it is a solution of the following social welfare maximization problem;

$$\text{Given } \lambda=(\lambda_2\ldots\lambda_m)\in\mathbb{R}^{m-1}_{++},$$
$$\text{maximize } u_1(\boldsymbol{x}_1)+\sum_{i=2}^{m}\lambda_i u_i(\boldsymbol{x}_i) \text{ subject to } \sum_i \boldsymbol{x}_i=\sum_i \omega_i. \tag{4}$$

It is easy to verify that the solution of this maximization problem exists. As will be shown in Proposition 1, the solution $\boldsymbol{x}_i$ associated with λ is a smooth function of $\lambda=(\lambda_2\ldots\lambda_m)$. Therefore we will denote them by $\boldsymbol{x}_i(\lambda)$.

On account of the second fundamental theorem of welfare economics, a Pareto optimal allocation which satisfies the budget constraints of all consumers is a competitive equilibrium. The following Negishi equation (Balasko [1997a]) then express this property of the competitive equilibrium.

$$\nu_i(\lambda,\omega)=Du_i(\boldsymbol{x}_i(\lambda))(\boldsymbol{x}_i(\lambda)-\omega_i)=0,\ \ i=2\ldots m. \tag{5}$$

Note that the first consumer's budget equation follows from the feasibility condition; $\sum_{i=1}^{m}(\boldsymbol{x}_i-\omega_i)=0$ and the first order conditions of the maximization problem, namely that $Du_1(\boldsymbol{x}_i)=\lambda_i Du_i(\boldsymbol{x}_i), i=2\ldots m$. See the proof of Proposition 1. The next definition is due to Balasko [1997a].

Definition 2: A pair $(\lambda,\omega)\in\mathbb{R}^{m-1}_{++}\times\Omega$ is an equilibrium if it is a solution of the Negishi equation systems

$$\nu_i(\lambda,\omega)=Du_i(\boldsymbol{x}_i(\lambda))(\boldsymbol{x}_i(\lambda)-\omega_i)=0,\ i=2\ldots m.$$

Example: There exists one consumer. The utility function is of *log*-linear form,

$$u(\boldsymbol{x})=\sum_{t=0}^{l}\delta^t log x^t,\ 0<\delta^t<1. \tag{6}$$

Then we have,

$$Du(\boldsymbol{x})=(\frac{1}{x^0}\ldots\frac{\delta^l}{x^l}), \tag{7}$$

and

$$D^2u(\boldsymbol{x})=-\begin{bmatrix} \frac{1}{(x^0)^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\delta}{(x^1)^2} & 0 & \cdots & 0 \\ \cdots\cdots & & & & \\ 0 & 0 & 0 & \cdots & \frac{\delta^l}{(x^l)^2} \end{bmatrix}. \tag{8}$$

Let $G\subset\mathbb{R}^{m-1}_{++}\times\Omega$ denote the set of equilibria. The projection map $\pi:G\to\Omega$ is the restriction to G of the natural projection $(\lambda,\omega)\to\omega$.

Proposition 1：The Negishi function defined by

$$\nu:\mathbb{R}^{m-1}_{++}\times\Omega\to\mathbb{R}^{m-1},\ (\lambda,\omega)\mapsto\nu(\lambda,\omega)=(\nu_2(\lambda,\omega)\ldots\nu_m(\lambda,\omega))$$

is of class C^1.

Proof：First we shall show that the map $\lambda\mapsto x_i(\lambda)<,\ i=2\ldots m$ is smooth. Note that the function is determined implicitly by the first order condition of the social welfare maximization problem with a resource constraint;

$$\sum_i \boldsymbol{x}_i-\sum_i\omega_i=0 \tag{9}$$

$$Du_1(\boldsymbol{x}_i)-\lambda_i Du_i(\boldsymbol{x}_i)=0,\ i=2\ldots m. \tag{10}$$

It is sufficient to check that the Jacobian matrix of the above equation system

$$\begin{bmatrix} I & I & I & \cdots & I \\ -D^2u_1 & \lambda_2 D^2u_2 & 0 & \cdots & 0 \\ -D^2u_1 & 0 & \lambda_3 D^2u_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ -D^2u_1 & 0 & 0 & \cdots & \lambda_m D^2u_m \end{bmatrix} \tag{11}$$

is invertible (Theorem A1). This matrix is equivalent to

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ -D^2u_1 & D^2u_1+\lambda_2 D^2u_2 & D^2u_1 & \cdots & D^2u_1 \\ -D^2u_1 & D^2u_1 & D^2u_1+\lambda_3 D^2u_3 & \cdots & D^2u_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -D^2u_1 & D^2u_1 & D^2u_1 & \cdots & D^2u_1+\lambda_m D^2u_m \end{bmatrix}. \tag{12}$$

Therefore it suffices to show that the matrix

$$M=\begin{bmatrix} D^2u_1+\lambda_2D^2u_2 & D^2u_1 & \cdots & D^2u_1 \\ D^2u_1 & D^2u_1+\lambda_3D^2u_3 & \cdots & D^2u_1 \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots \\ D^2u_1 & D^2u_1 & \cdots & D^2u_1+\lambda_mD^2u_m \end{bmatrix} \tag{13}$$

is invertible. Let $v=(v_2\ldots v_m)\in(\mathbb{R}^l)^{m-1}$ and consider the equation $Mv=0$. Then we have

$$(D^2u_1+\lambda_jD^2u_j)v_j+D^2u_1\Big(\sum_{k\neq j}v_k\Big)=0,\ j=2\ldots m. \tag{14}$$

Making the inner product with the vector v_j then yields

$$\Big(v_j,(D^2u_1+\lambda_jD^2u_j)v_j\Big)+\Big(v_j,D^2u_1\Big(\sum_{k\neq j}u_k\Big)\Big)=0,\ j=2\ldots m, \tag{15}$$

which simplifies into

$$(\lambda_jv_j,D^2u_jv_j)+\Big(v_j,D^2u_1\Big(\sum_{k=2}^{m}u_k\Big)\Big)=0,\ j=2\ldots m. \tag{16}$$

We add up these equations for i from 2 to m,

$$\sum_{j=2}^{m}\lambda_j(v_j,\ D^2u_jv_j)+\Big(\Big(\sum_{j=2}^{m}v_j\Big),D^2u_1\Big(\sum_{j=2}^{m}v_j\Big)\Big)=0. \tag{17}$$

Since D^2u_j are negative definite, each term in the sum of the left hand side of (14) is ≤ 0, hence each term must be 0,

$$\Big(\Big(\sum_{j=2}^{m}v_j\Big),\ D^2u_1\Big(\sum_{j=2}^{m}v_j\Big)\Big)=0,\ \lambda_j(v_j,D^2u_jv_j)=0,\ j=2\ldots m. \tag{18}$$

Therefore we have $v_2=\cdots=v_m=0$ as desired. (Q. E. D)

Therefore by construction, the Negishi map

$$\nu_i((\lambda_i),(\omega_i))=Du_i(\boldsymbol{x}_i(\lambda))(\boldsymbol{x}_i(\lambda)-\omega_i)$$

is of class C^1 with respect to $(\lambda,\omega)=((\lambda_i),(\omega_i))$.

Proposition 2: The set of equilibria G is a smooth submanifold of $\mathbb{R}^{m-1}\times\Omega$ of codimension $m-1$.

Proof: We shall prove this proposition by applying the regular value theorem (Theorem A2, appendix), which amounts to prove that $0\in\mathbb{R}^{m-1}$ is a regular value of the map ν, which is smooth by Proposition 1. Let $D\nu(\lambda,\omega):\mathbb{R}^{m-1}\times T_\omega\Omega\rightarrow\mathbb{R}^{m-1}$ denote the tangent map (derivative) of map ν at $(\lambda,\nu)\in G$, where $T_\omega\Omega$ is the tangent space of Ω at ω. We denote the tangent map $D\nu(\lambda,\omega)$ as $(D\nu_i(\lambda,\omega))_{i=2}^{m}$, which is defined by the $m-1$ coordinate mappings of the derivatives of ν_i with respect to $(\lambda,\omega_i\ldots,\omega_m)$. Note that the partial derivative of ν_i with respect to ω_j $(j\neq i)$ is 0. The derivative of ν_i is then written as

$$D\nu_i(\lambda,\omega_i)(\dot{\lambda},\dot{\omega}_i)=\sum_{j=2}^{m}\frac{\partial\nu_i(\lambda,\omega_i)}{\partial\lambda_j}\dot{\lambda}_i+\sum_{t=1}^{m}\frac{\partial\nu_i(\lambda,\omega_i)}{\partial_t\omega_i^t}\dot{\omega}_i^t, \tag{19}$$

where $\dot{\lambda}\in\mathbb{R}^{m-1}$ and $\dot{\omega}_i\in\mathbb{R}^{m-1}$ are tangent vectors. In order to show that the $D\nu(\lambda,\omega)$ is onto, it suffices to prove that a restriction of this map to some subspace of $\mathbb{R}^{m-1}\times T_\omega\Omega$ is onto. Pick an arbitrary period t. The Jacobian matrix of $D\nu$ with respect to $(\omega_2^t\ldots\omega_m^t)$ is

$$\begin{bmatrix} \partial_t u_2(\boldsymbol{x}_2(\lambda)) & 0 & \cdots & 0 \\ 0 & \partial_t u_3(\boldsymbol{x}_3(\lambda)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \partial_t u_m(\boldsymbol{x}_m(\lambda)) \end{bmatrix}, \tag{20}$$

which is obviously of rank $m-1$ by the assumption (U-2). (Q. E. D.)

By Proposition 2, it follows that the projection map $\pi: G\to\Omega$, $(\lambda,\omega)\mapsto\omega$ is smooth, since it is a composition of smooth maps, the canonical embedding $G\to\mathbb{R}^{m-1}_{++}\times\Omega$ and the canonical projection map $\mathbb{R}^{m-1}_{++}\times\Omega\to\Omega$.

We now come up with the definition of our main theme of this paper.

Definition 3：a λ-equilibrium $(\lambda,\omega)\in G$ is called regular if it is a regular point of the projection map $\pi: G\to\Omega$. The regular value $\omega\in\Omega$ is called a regular economy. An economy which is not regular is called critical.

An important property of the map π is the following.

Proposition 3：The projection map $\pi: G\to\Omega$ is proper, that is, its inverse $\pi^{-1}(K)$ of every compact set $K\subset\Omega$ is compact.

Proof：Let K be a compact subset of Ω and K_i be the image of K by the restriction of the natural projection map $(\boldsymbol{x}_i)\to\boldsymbol{x}_i$. The set K_i is therefore a compact subset of $X_i=\mathbb{R}^l_{++}$. The utility function $u_i: X_i\to\mathbb{R}$ being smooth, hence continuous, so that the image $u_i(K_i)$ is compact, hence bounded by some $b_i\in\mathbb{R}$, $i=1\ldots m$. Define a subset L of $\mathbb{R}^{lm}$ by

$$L=\{(\boldsymbol{x}_i)\in\mathbb{R}^{lm}_{++}|\sum_i\boldsymbol{x}_i=\sum_i\omega_i,\ u_i(\boldsymbol{x}_i)\geq b_i,\ i=1\ldots m\}.$$

We claim that the set L is compact. First for each $i=1\ldots m$, define the set L_i by $L_i=\{\boldsymbol{x}_i\in\mathbb{R}^m_{++}|\boldsymbol{x}_i\leq\sum_i\omega_i, u_i(\boldsymbol{x}_i)\geq b_i\}$. The set L_i is bounded since $\mathbf{0}$ is a lower bound of L_i and $\sum_i\omega_i$ is an upper bound. Next the set L_i is closed. For let $(\boldsymbol{x}_n)$, $n=1,2,\ldots$ be a sequence in L_i with $\boldsymbol{x}_n\to\boldsymbol{x}_*$. Clearly $\boldsymbol{x}_*\leq\sum_{i=1}^m\omega_i$ and by the continuity of u_i, we have $u_i(\boldsymbol{x}_*)\geq b_i$. It remains to show that $\boldsymbol{x}_*\in\mathbb{R}^l_{++}$. Obviously $\boldsymbol{x}_*\geq 0$. Since $u_i(\boldsymbol{x}_*)\geq b_i>0$, it is impossible that $\boldsymbol{x}_*^t=0$ for any t by assumption (U-2). Therefore L_i is compact, hence the product $\prod_i L_i$ is also compact by the Tychonoff theorem. As a closed subset of $\prod_i L_i$, the set L is compact.

The individual rationality means that if $\boldsymbol{x} \in \Omega$ is an equilibrium allocation associated with $\omega \in \Omega$, then the inequality $u_i(\boldsymbol{x}_i) \geq u_i(\omega_i)$ is satisfied for all $i=1 \ldots m$. This implies that all equilibrium allocations $(\boldsymbol{x}_i)$ associated with endowments $\omega \in K$ satisfy the inequality $u_i(\boldsymbol{x}_i) \geq u_i(\omega_i) \geq b_i$ is satisfied for all $i=1 \ldots m$. In other words, if $(\omega_i) \in K$, then $(\boldsymbol{x}_i) \in L$. We now claim that the set of Pareto optimal allocations associated with endowments $(\omega_i) \in K$ is a closed subset of a compact set L, hence compact.

Let $(\boldsymbol{x}_i^n), n=1,2,\ldots$ be a sequence of the solutions of social welfare maximization problem associated with the endowments (ω_i^n) in K such that $\boldsymbol{x}_i^n \to \boldsymbol{x}_i^*$. Then for each n, there exists a social welfare weight $(\lambda_i^n)_{i=2}^m$ which satisfies the first order conditions

$$\sum_i x_i^n - \sum_i \omega_i^n = 0, \tag{21}$$

$$Du_1(\boldsymbol{x}_i^n) - \lambda_i^n Du_i(\boldsymbol{x}_i^n) = 0,\ i=2 \ldots m,\ n=1,2,\ldots. \tag{22}$$

Since K is compact, we may assume $\omega_i^n \to \omega_i^*$, $i=2 \ldots m$. For each n, we have $\lambda_i^n = \partial_1 u_1(x_1^n)/\partial_1 u_i(x_i^n)$. Since the partial derivative are continuous and $x_i^n \to x_i^*$, it follows that $\lambda_i^n \to \lambda_i^* = \partial_1 u_1(x_1^*)/\partial_1 u_i(x_i^*)$ by the assumption (U-2). Clearly one has

$$\sum_i x_i^* - \sum_i \omega_i^* = 0, \tag{23}$$

$$Du_1(\boldsymbol{x}_1^*) - \lambda_i^* Du_i(\boldsymbol{x}_i^*) = 0,\ i=2 \ldots m,\ n=1,2,\ldots. \tag{24}$$

This shows that the set of Pareto optimal allocations associated with endowments in K is a closed subset of L, as desired. The set of associated welfare weights is then necessarily a compact subset H of $\mathbb{R}_{++}^{m-1}$. Since $\pi^{-1}(K) = (H \times K) \cap G$ and G is closed in $\mathbb{R}^{m-1} \times \Omega$, it follows that $\pi^{-1}(K)$ is compact. (Q. E. D.)

The next proposition which characterizes the regular and/or singular equilibria will be useful for the subsequent analysis.

Proposition 4: The λ-equilibrium $(\lambda, \omega) \in G$ is critical if and only if

$$det \frac{D\nu(\lambda, \omega)}{D\lambda} = 0,$$

where $det\ A$ means the determinant of a matrix A.

Proof: The tangent space of G at (λ, ω) and the derivative map of the projection π are written as

$$T_{(\lambda,\omega)} G = \left\{ (\dot{\lambda}, \dot{\omega}) \in \mathbb{R}^{m-1} \times \mathbb{R}^{lm} \,\middle|\, \frac{D\nu(\lambda,\omega)}{D\lambda} \dot{\lambda} + \frac{D\nu(\lambda,\omega)}{D\omega} \dot{\omega} = 0 \right\},$$

and

$$D\pi : T_{(\lambda,\omega)}G \to \mathbb{R}^{lm}, (\dot{\lambda}, \dot{\omega}) \mapsto \dot{\omega},$$

where $\dot{\lambda} \in \mathbb{R}^{m-1}$ and $\dot{\omega} \in \mathbb{R}^{lm}$ are the tangent vectors.

Since $dimention\, T_{(\lambda,\omega)}G = lm$, the map $D\pi$ is onto if and only if the linear subspace of $T_{(\lambda,\omega)}G$ of the form $\{(\dot{\lambda}, \dot{\omega}) \in T_{(\lambda,\omega)}G \mid \dot{\omega} = \mathbf{0}\}$ contains the linear space other than $\{0\}$. The necessary and sufficient condition for this is that the linear equation $\frac{D\nu(\lambda,\omega)}{D\lambda}\dot{\lambda} = 0$ has a nonzero solution $\dot{\lambda} \neq 0$, namely that $det\frac{D\nu(\lambda,\omega)}{D\lambda} = 0$. (Q. E. D.)

The fundamental properties of the regular economy is that they have finitely many (regular) equilibria which are locally stable and unique.

Theorem 1： If ω is a regular economy, then the set of λ-equilibria associated with ω is finite.

Proof： By Proposition 4, a λ-equilibrium $(\lambda_0, \omega_0) \in G$ is regular if and only if $det\frac{D\nu(\lambda,\omega)}{D\lambda} \neq 0$. Then by the implicit function theorem (Theorem A1), we can take a neighborhood U_0 of ω_0 such that λ is a C^1 function of ω and $\nu(\lambda(\omega), \omega) = 0$ for all $\omega \in U_0$. Let $G_0 = \{(\lambda, \omega) \in G \mid \omega \in U_0\}$ and define the map $\rho : U_0 \to G_0$ by $\rho(\omega) = (\lambda(\omega), \omega)$. We then have $\pi \circ \rho =$ identity on U_0, which means that π restricted to G_0 is a diffeomorphism between U_0 and G_0 with the inverse ρ.

Let ω be regular. Since π is a proper map by Proposition 3, the set $\pi^{-1}(\omega)$ is compact. In order to show that the set $\pi^{-1}(\omega)$ is finite, it suffices to show that $\pi^{-1}(\omega)$ is discrete. Take open sets U_0 and G_0 of the preceding discussion. We claim that $\pi^{-1}(\omega) \cap G_0$ is one point set $\{(\lambda(\omega), \omega)\}$. If not, $\{(\lambda(\omega), \omega)\}$ contains at least two distinct points both of which are mapped to ω by the projection π. This contradicts the fact that π restricted to G_0 is bijective. (Q. E. D.)

Let R be the set of regular economies. We will show in Theorem 3 that R is open subset of $\Omega = \mathbb{R}^{lm}_{++}$. The next theorem shows that the regular equilibrium is locally unique and moves continuously when the economy changes continuously (locally stable).

Theorem 2：For every $\omega \in$ R, there exists an open neighborhood V of ω in R such that the preimage $\pi^{-1}(\omega)$ is the disjoint union of a family of open subsets U_n of π^{-1}(R) and the restriction π_n : $U_n \to V$ of π to each U_n is a homeomorphism.

Proof：By Theorem 2, $\pi^{-1}(\omega)$ is a finite set $\{(\lambda_1, \omega) \dots (\lambda_k, \omega)\}$. As in the proof of Theorem 1, we can take open disjoint neighborhoods $U'_1 \dots U'_k$ of regular equilibria $(\lambda_1, \omega) \dots (\lambda_k, \omega)$ such that the restriction of π to U'_i is a diffeomorphism with $V_i = \pi(U'_i)$, $i = 1 \dots k$.

We claim that the image of a closed set by the proper map π is closed. For let C be a closed set and take a converging sequence $\{y_n\}$ in $\pi(C)$, $y_n \to y_*$. The set $Y = \{y_n\} \cup \{y_*\}$ being compact, $\pi^{-1}(Y)$ is compact subset of C by the properness of π. Take $x_n \in \pi^{-1}(y_n)$ for each n. Then we can assume that $x_n \to \boldsymbol{x}_* \in \pi^{-1}(Y) \subset C$. Since the projection is continuous, we have $\boldsymbol{y}_* = \pi(\boldsymbol{x}_*) \in \pi(C)$, as desired.

Since the set $G\backslash(U'_1\cup\ldots\cup U'_k)$ is closed in G, its image by the map π is closed. Define the set V by

$$V=(V_1\cap\ldots\cap V_k)\backslash\pi(G\backslash(U'_1\cup\ldots\cup U'_k)).$$

Clearly the set V is open in Ω. We show that $\omega\in V$. Since $\omega\in\cap_{i=1}^{k}V_i$, it suffices to show that $\omega\notin\pi(G\backslash(U'_1\cup\ldots\cup U'_k))$. This follows from the fact that $\pi^{-1}(\omega)\subset U'_1\cup\ldots\cup U'_k$. Define $U_n=U'_n\cap\pi^{-1}(V)$. The restriction $\pi_n=\pi|U_n$ is then a homeomorphism between U_n and $\pi(U_n)=V$. It remains to prove that $\pi^{-1}(V)=\cap_{n=1}^{k}U_n$. Suppose not. Then there exists $x'\in\pi^{-1}(V)$ such that $x'\notin U_n$ for some n. Then x' must belong to $G\backslash(U'_1\cup\ldots\cup U'_k)$, which implies that $\omega'=\pi(x')\in\pi(G\backslash(U'_1\cup\ldots\cup U'_k))$. Therefore $\omega'\notin V$, contradicting the choice of $x'\in\pi^{-1}(V)$.

Finally we prove that the size of the regular economies is "big" in the space of all economics, in other words, the size of singular economies is "small" in the set Ω.

Theorem 3: The set of singular economies is a closed subset of Ω which has measure zero.

Proof: It follows from theorem A3 in Appendix that the set of critical economies, as the set of critical values of a smooth map $\pi:G\to\Omega$, is of measure zero, since dimension $G=$dimension Ω. By Proposition 4, the set of critical equilibria is a closed subset of G, since the function $det\frac{D\nu(\lambda,\omega)}{D\lambda}$ is continuous. The set of critical economies is therefore closed, since it is a the image of a proper map π. (Q. E. D.)

Appendix

In this appendix, we collect mathematical concepts and results which are used in the text for reader's convenience.

Let E and F be Banach spaces and $U\subset E$ be open. A map $f:U\to F$ is said to be linear if

$$\phi(a\boldsymbol{x}+b\boldsymbol{y})=a\phi(\boldsymbol{x})+b\phi(\boldsymbol{y}) \tag{25}$$

for every $\boldsymbol{x},\boldsymbol{y}\in E$ and every $a,b\in\mathbb{R}$. This idea of linearity to the idea of multi-linear map. Let $E_1\ldots E_n$, F be Banach spaces. A map $\psi:E_1\times\ldots\times E_n\to F$ is said to be n-multi-linear if $\psi(x_1\ldots x_n)$ is linear in each variable separately. For instance, the linearity in the first variable means that

$$\psi(a\boldsymbol{x}_1+b\boldsymbol{y}_1\ldots\boldsymbol{x}_n)=a\phi(\boldsymbol{x}_1\ldots\boldsymbol{x}_n)+b\phi(\boldsymbol{y}_1\ldots\boldsymbol{x}_n). \tag{26}$$

The space of n-multi-linear maps of $E_1\ldots E_n$ to F is denoted by $L(E_1\ldots E_n,F)$. When $E_1=\ldots=E_n=E$, it is denoted by $L^n(E,F)$. In particular for $k=1$, we usually write $L^1(E,F)=L(E,F)$. This is nothing but the space of linear maps of E to F. Banach spaces E and F are said to be isomorphic if there exists a bijective (namely one to one and onto) and continuous map f of E to F whose inverse f^{-1} is also continuous. Two isomorphic Banach spaces are considered to be the same space. Hence if E and F are isomorphic, we often write $E=F$.

In the following, we assume that $E_1 \dots E_n=\mathbb{R}^k$ and $F=\mathbb{R}^l$. Therefore every element ψ of $L^n(E,F)$ is continuous and we can identify the space of linear maps of $E=\mathbb{R}^k$ to $F=\mathbb{R}^l$ or $L(E,F)$ with $\mathbb{R}^{kl}$, since a point ψ in $L(E,F)$ has kl "coordinates" $\psi^j(\boldsymbol{e}^i)$, $i=1 \dots k$, $j=1 \dots l$, where $\boldsymbol{e}^i=(0 \dots 1 \dots 0)$ with 1 in i-th position. Similarly, we can identify $L_n(E,F)$ with $\mathbb{R}^{knl}$. Moreover, we can show that $L(E,L_{n-1}(E,F))=L^n(E,F)$. Indeed. let $\phi \in L(E,L^{n-1}(E,F))$. Since $\phi(\boldsymbol{x}_1) \in L^{n-1}(E,F)$, we can define a map $\psi:E^n \rightarrow F$ by $\psi(\boldsymbol{x}_1 \dots \boldsymbol{x}_n)=\phi(\boldsymbol{x}_1)(\boldsymbol{x}_2 \dots \boldsymbol{x}_n)$. Obviously the map $\phi \mapsto \psi$ is bijective and linear, hence $L(E,L^{n-1}(E,F))$ and $L^n(E,F)$ are isomorphic.

Let U be an open subset of $E=\mathbb{R}^k$. A map $f:U \rightarrow F(=\mathbb{R}^l)$ is called differentiable at $x \in U$ if there is a continuous linear map $Df(x) \in L(E,F)$ such that

$$\lim_{h \rightarrow 0} \frac{||f(\boldsymbol{x}+\boldsymbol{h})-f(x)-Df(\boldsymbol{x})||}{||\boldsymbol{h}||}=0. \tag{27}$$

The linear map $Df(\boldsymbol{x}) \in L(E,F)$ is called the derivative of f at $\boldsymbol{x}$. An advantage of this definition of the derivative is "coordinate free", hence it can be applied to the case of general Banach spaces. When we use the coordinates of $E=\mathbb{R}^k$ and $F=\mathbb{R}^l$, $f(\boldsymbol{x})=(f^1(\boldsymbol{x}) \dots f^l(\boldsymbol{x}))$, the derivative can be written in the standard matrix form;

$$Df(\boldsymbol{x})=\begin{bmatrix} \partial_1 f^1(\boldsymbol{x}) & \cdots & \partial_l f^1(\boldsymbol{x}) \\ \cdots & \cdots & \cdots \\ \partial_1 f^l(\boldsymbol{x}) & \cdots & \partial_l f^l(\boldsymbol{x}) \end{bmatrix}. \tag{28}$$

This $k \times l$ matrix is often called the Jacobian matrix of f.

For every integer $r \geq 0$, the r-th derivative $D^r f(\boldsymbol{x})$ of f at $x \in U$ is defined inductively

$$D^r f(\boldsymbol{x}) \equiv D(D^{r-1}f)(\boldsymbol{x}):U \rightarrow L(E,L^{r-1}(E,F)) \equiv L^r(E,F) \tag{29}$$

which maps $\boldsymbol{x}$ at U to an n-multi-linear map of E to F. A map f is said to be of class C^r at $\boldsymbol{x} \in U$ if this map is continuous. It is equivalent with that every partial derivative (in the usual sense) $\frac{\partial^n f^j}{\partial x_{i1} \dots \partial x_{in}}(\boldsymbol{x}), 1 \leq i^1 \dots i^n \leq k, 1 \leq j \leq l$, exists and continuous.

We can also give a "coordinate free" definition of the partial derivative. Let $E = E_1 \oplus E_2$ (direct sum), where E_1 and E_2 are Banach spaces, and $U \subset E$ is open subset of E. For a map of class C^r $f:U \rightarrow F$, the partial derivative with respect to E_1 is defined by

$$\partial_1 f(\boldsymbol{x}) \equiv Df(\boldsymbol{x})(e_1,0),\ e_1 \in E_1. \tag{30}$$

The partial derivative with respect to E_2 is defined similarly. The next theorem is of central importance in the differential calculus on manifolds (Abraham et al [1981, p. 121]).

Theorem A1 (Implicit function theorem) : Let $U \subset E, V \subset F$ be open and $f:U \times V \rightarrow F$ be of class $C^r(r \geq 1)$.

For some $\boldsymbol{x}_0 \in U$ and $\boldsymbol{y}_0 \in V$, assume that $\partial_2 f(\boldsymbol{x}_0,\boldsymbol{y}_0):F \rightarrow F$ is an isomorphism. Then there are neighborhoods U_0 of $\boldsymbol{x}_0$ and $W \subset F$ of $f(\boldsymbol{x}_0, \boldsymbol{y}_0)$, and a unique C^r map $g: U_0 \times W \to V$such that for all $(\boldsymbol{x}, \boldsymbol{z}) \in U \times W$,

$$f(\boldsymbol{x}, g(\boldsymbol{x}, \boldsymbol{z})) = \boldsymbol{z}.$$

We now give the definition of the differentiable manifold.

Definition A1 : A (Hausdorrf) topological space M is a k-dimensional manifold if there exist an open cover $\{U_\alpha\}$ and local isomorphisms ϕ_α on U_α to M such that $\phi_\beta \circ \phi_\alpha^{-1}: \phi_\alpha(U_\alpha \cap U_{\beta,}) \to \phi_\beta(U_\alpha \cap U_\beta)$ is bijective and of class C^r for each α and β.

Since $(\phi_\beta \circ \phi_\alpha^{-1})^{-1} = \phi_\alpha \circ \phi_\beta^{-1}: \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$, the inverse of $\phi_\beta \circ \phi_\alpha^{-1}$ is also of class C^r, so that it is C^r-diffeomorphism. (U_α, ϕ_α) is called the chart and the family of all charts is called an atlas.

Let $f: M \to N$ be a map from a manifold M to a manifold N. The map f is said to be at class C^r, if the map $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ from $\phi_\alpha(U_\alpha)$ to $\psi_\beta(V_\beta)$ is of class C^r, where (U_α, ϕ_α) and (V_β, ψ_β) are charts of M and N, respectively. This definition does not depend on the choice of the coordinate charts. Indeed, let (U_γ, ϕ_γ) and (V_δ, ψ_δ) be another charts of M and N. Since $\psi_\delta \circ f \circ \phi_\gamma^{-1} = (\psi_\delta \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ f \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi_\gamma^{-1})$ and $\psi_\delta \circ \psi_\beta^{-1}$ and $\phi_\alpha \circ \phi_\gamma^{-1}$ are of class C^r by definition, $\psi_\delta \circ f \circ \phi_\gamma^{-1}$ is of class C^r. As a special case, a C^r curve through $p \in M$ is a C^r map from $(-\epsilon, \epsilon)$ to M such that $c(0) = p$. Two C^r curves c and d are equivalent if and only if $\dot{c}(0)\,(\equiv d(\phi_\alpha \circ c)/dt(0)) = \dot{d}(0)$, where (U_α, ϕ_α) is a chart of M such that $p \in U_\alpha$. This definition of the equivalence relation is also independent of the choice of the coordinate chart. Let (U_β, ϕ_β) be another chart with $p \in U_\beta$. From $\phi_\beta \circ c = (\phi_\beta \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ c)$ and $\phi_\beta \circ d = (\phi_\beta \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ d)$, it follows that $d(\phi_\beta \circ c)/dt = D(\phi_\alpha \circ \phi_\alpha^{-1}) d(\phi_\alpha \circ c)/dt$ and $d(\phi_\beta \circ d)/dt = D(\phi_\beta \circ \phi_\alpha^{-1}) d(\phi_\alpha^{-1} \circ d)/dt$. Since $D(\phi_\beta \circ \phi_\alpha^{-1})$ is a linear isomorphism, $d(\phi_\beta \circ c)/dt = d(\phi_\beta \circ d)/dt$ if and only if $d(\phi_\alpha \circ c)/dt = d(\phi_\alpha \circ d)/dt$.

The equivalence class is denoted as $[c]_p$ and called a tangent vector at p. Let T_pM be the set of all tangent vectors at p and call the tangent space at p. It is easy to see that T_pM is a k-depersonal vector space. Let (U_α, ϕ_α) be a coordinate chart with $p \in U_\alpha$. Without loss of generality, we may assume that $\phi(p) = \mathbf{0} \in \mathbb{R}^k$. Define the smooth curves $c_i(t) = \phi^{-1}(0, \ldots, t, \ldots, 0)$, where t is at the i-th coordinate. Then the curves $[c_i]_p$, $i = 1 \ldots k$, make up with a basis of T_pM.

Let f be a C^r map from a manifold M to a manifold N. The tangent map (derivative) at $p \in M$ of a C^r map f is a linear map $Df(p): T_pM \to T_{f(p)}N$ defined by

$$Df(p)([c]_p) = [f \circ c]_{f(p)}.$$

We have to check that this definition is independent of the choice of curves representing the equivalence class. Let c_1 and c_2 be two curves such that $[c_1]_p = [c_2]_p$. This means that $d(\phi_\alpha \circ c_1)/dt = d(\phi_\alpha \circ c_2)/dt$, where (U_α, ϕ_α) is a chart on M with $p \in U_\alpha$. We want to show that $[f \circ c_1]_{f(p)} = [f \circ c_2]_{f(p)}$. Since $\psi_\gamma \circ f \circ c_i = (\psi_\gamma \circ f \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ c_i)$, $i = 1, 2$, where (V_γ, ψ_γ) is a chart on N with $f(p) \in V_\gamma$, one has $d(\psi_\gamma \circ f \circ$

$c_i)/dt = D(\psi_\gamma \circ f \circ \phi_\alpha^{-1}) d(\phi_\alpha \circ c_i)/dt$, $i=1,2$. Since $d(\phi_\alpha \circ c_1)/dt = d(\phi_\alpha \circ c_2)/dt$, we get $d(\psi_\gamma \circ f \circ c_1)/dt(0) = d(\psi_\gamma \circ f \circ c_2)/dt(0)$.

Let $f: M \to N$ be a smooth (C^r) map between manifolds M and N. A point $q \in N$ is a regular value of f (or f is transversal to $\{q\}$) if for every $p \in f^{-1}(q)$, $Df(p): T_pM \to T_qN$ is surjective (onto). Note that when dimension $M <$ dimension N, a point q is regular value only if $q \notin f(M)$. When dimension $M =$ dimension N, $q \in N$ is regular value if and only if $Df(p)$ is isomorphism between T_pM and T_pN at every $p \in f^{-1}(q)$. A point of N which is not regular point is called a critical point.

Theorem A2 (Regular value theorem): Let $f: M \to N$ be a smooth (C^r) map between manifolds M and N such that dimension $M \geq$ dimension N and $q \in N$ a regular value of f. Then $f^{-1}(q)$ is a submanifold of M such that dimension $f^{-1}(q) =$ dimension $M -$ dimension N

A subset R of $\mathbb{R}^l$ is called a rectangular solid if it is of the form $R = \{(x^1 \ldots x^l) | a^h \leq x^h \leq b^h, h = 1 \ldots l\}$ for vectors $a = (a^1 \ldots a^l)$ and $b = (b^1 \ldots b^l)$ with $a^h < b^h$ for all $h = 1 \ldots l$. The volume of the rectangular solid R is defined by

$$volR = \prod_{h=1}^{l} (b^h - a^h).$$

A subset $A \subset \mathbb{R}^l$ is said to have measure zero if for every $\epsilon > 0$, there exist countably many rectangular solids $R_1, R_2 \ldots$ such that $A \subset \cup_{j=1}^{\infty} R_j$ and $\sum_{j=1}^{\infty} volR_j < \epsilon$.

The next theorem is a key of the analysis of the text.

Theorem A3 (Sard): Let $f: M \to V$ be a C^r map where M is a manifold of dimension k and V is an open subset of $\mathbb{R}^l$ (hence a manifold of dimension l). If $k \geq l$ and $r > max\{0, k-l\}$, then the set of regular values of f has measure zero.

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