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On the Continuity of Convex Correspondences*

Jinryo Takasaki

Abstract

We prove that a compact-valued correspondence with strictly convex graph is continuous and that a compact-valued correspondence with convex graph is continuous on the interior of its domain. Some other results are useful in order to confirm, for example, the lower hemi-continuity of a correspondence whose graph is ‘not necessarily convex’ and have wide applicability to economic theory.

Key Words : correspondence, continuity, convexity, interior point, induced topology.

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1. Introduction

The first purpose of this paper is to prove that a correspondence with convex graph is lower hemi-continuous on the interior of its domain and that a correspondence with strictly convex graph is lower hemi-continuous (proposition 2.1). This proposition has some useful byproducts. In order to confirm the lower hemi-continuity of a correspondence whose graph is ‘not necessarily convex’, our proposition 2.2 is useful. For example, as a corollary of proposition 2.2 we get a result that any corre-

＊ This paper is a revised version of Takasaki 1991.

spondence with open graph is lower hemi-continuous. At a boundary point of the domain we can not prove the lower hemi-continuity of a correspondence with convex graph because we have a counter example (c.f. the example in section 2).

Our second purpose is to prove that a compact-valued correspondence with strictly convex graph is upper hemi-continuous (proposition 3.1), and that a compact-valued correspondence with convex graph is upper hemi-continuous on the interior of its domain (proposition 3.2). At a boundary point of the domain we can not prove the upper hemi-continuity of a correspondence with convex graph because we have a counter example (c.f. figure 1 in section 3).

From these propositions we finally get the results that a compact-valued correspondence with strictly convex graph is continuous (proposition 4.1) and a compact-valued correspondence with convex graph is continuous on the interior of its domain (proposition 4.2).

In section 5, we refer to some applications of our results to economics. It seems that our propositions have wide applicability to economic theory which has many convex structures.

2. Lower Hemi-Continuity

We use the term 'correspondence' as a 'nonempty-valued relation', i.e. we assume that $f(x) \neq \phi$ for all $x \in X$.

Definition 2.1.1 Let X, Y be subsets of metric spaces, and $x' \in X$. A correspondence f: X → Y is 'lower hemi-continuous (and we abbreviate this to l.h.c.) at x′' if, for every $y' \in f(x')$ and for every sequence $\{x_n\}_{n \in N}$ in X which converges to x′, there exists a sequence $\{y_n\}_{n \in N}$ which converges to y′ with $y_n \in f(x_n)$ for all $n \in N$. If f is l.h.c. at every point of X, we say that 'f is l.h.c.'. (Stokey and Lucas 1989).

Definition 2.1.2 Let X, Y be subsets of topological spaces, and $x' \in X$. A correspondence f: X → Y is 'lower hemi-continuous (and we abbreviate this to l.h.c.) at x′' if, for every open set U in Y with $f(x') \cap U \neq \phi$, there exists a neighborhood V of x′ such that $f(x) \cap U \neq \phi$, for all $x \in V$. If f is l.h.c. at every point of X, we say that 'f is l.h.c.'. (Berge 1963 or Hildenbrand 1974).

Definition 2.2 For a correspondence f: X → Y, the graph of f means the set G(f).

$$G(f) \equiv \{(x, y) \mid x \in X, y \in f(x)\}$$

Since a correspondence f: X → Y can be defined as a subset of product X × Y, there is no essential difference between G(f) and f itself. But, in order to appeal to geometrical image of convexity, we use the term 'graph of a correspondence'.

Definition 2.3 Let A be a subset of a linear metric space, A^a be the closure of A and A^i be the interior of A. If the set

$$A' \equiv \{\lambda x + (1-\lambda)\, y \mid x, y \in A^a, 0 < \lambda < 1, x \neq y\}$$

is contained in A^i, we say that 'A is strictly convex'.

We prove the following proposition.

Proposition 2.1 We consider sets $X \subset R^m$, $Y \subset R^n$ and a correspondence f: X → Y (Here, R^m is the m dimensional real space).

1) If the graph of f, G(f), is convex, then f is l.h.c. at any point in the interior of X.

2) If G(f) is strictly convex, then f is l.h.c.

Proof Let us consider $(x', y') \in G(f)$. For $\varepsilon > 0$ and for $i \in N$, we define B_i, A_i and P_i as follows.

$$B_i \equiv \{(x, y) \mid d((x', y'), (x, y)) < \varepsilon/i, x \in X \text{ and } y \in Y\}$$
$$A_i \equiv B_i \cap G(f)$$
$$P_i \equiv \{ x \mid \exists y: (x, y) \in A_i \}$$

Since A_i is the intersection of convex sets, A_i is convex. P_i is also convex because it is a projection of a convex set.

Proof of 1) First, we show that if x′ is an interior point of X, then it is also an interior point of P_i by the induced topology of X.

We suppose that x′ is a boundary point of P_i, then, by the separation theorem, there exists a supporting hyperplane H and the supporting halfspace S, with the following properties:

$$x' \in H,\ H \subset S,\ P_i \subset S \text{ and } X-S \text{ is not empty and open.}$$

Let us pick out arbitrary $x_1 \in X-S$ and $y_1 \in f(x_1)$, and define the sequence of (x_n, y_n) as follows:

$$x_n = (1-1/n)\, x' + x_1/n$$
$$y_n = (1-1/n)\, y' + y_1/n$$

Here we remark that $x_n \in X-S$ for any $n \in N$. Since $(x_1, y_1) \in G(f)$, and since (x_n, y_n) is a convex combination of (x', y') and (x_1, y_1), it follows that $(x_n, y_n) \in G(f)$ for any $n \in N$. There exists a sufficiently large number M such that $(x_n, y_n) \in B_i$ for $n \geq M$. Therefore $(x_n, y_n) \in A_i$ and $x_n \in P_i$ for $n \geq M$. On the other hand, we know $x_n \in X - S$ for any $n \in N$. This is a contradiction.

From the above fact, we know that, for any $i \in N$, x′ is an interior point of P_i by the induced topology of X. It is easily seen that $P_{i+1} \subset P_i$. Let $\{ x_m \}_{m \in N}$ be a sequence in X which converges to x′. For any i, we can take a large number m(i) with the property ; '$x_m \in P_i$ for any $m \geq m(i)$ '. We can choose such a function, m: N → N, satisfying that $m(i) \leq m(i+1)$ (the axiom of choice which we assume all over this paper). Since $x_{m(i)} \in P_i$, there exists $y_{m(i)}$ such that $(x_{m(i)}, y_{m(i)}) \in A_i \subset B_i$. Then the sequence $\{(x_{m(i)}, y_{m(i)})\}_{i \in N}$ converges to (x′, y′).

Since $x_m \in P_i$ for m with $m(i) < m < m(i+1)$, there exists y_m such that $(x_m, y_m) \in A_i \subset G(f)$ for $x_m \in P_i$. Now we have constructed the sequence $\{(x_m, y_m)\}_{m \in N}$ in G(f) that converges to (x′, y′). Therefore the sequence $\{ y_m \}_{m \in N}$ converges to y′ and $y_m \in f(x_m)$ for all $m \in N$.

Proof of 2) Now, it is sufficient to consider only boundary points of X. Let us assume that x′ is a boundary point of X and $(x', y') \in G(f)$.

First we prove that f(x′) is a singleton (a set which has only one element). Suppose there exists an element $y'' \in f(x')$ with $y'' \neq y'$. Then, for λ with $0 < \lambda < 1$,

$$z \equiv \lambda (x', y') + (1 - \lambda) (x', y'')$$

is an interior point of G(f) by the strict convexity of G(f). A ball with center z and sufficiently small radius $\varepsilon > 0$,

$$B(z) \equiv \{ (x, y) \mid d (z, (x, y)) < \varepsilon, (x, y) \in R^{m+n} \},$$

is contained in G(f). Therefore the projection of B(z) into R^m,

$$P(z) \equiv \{ x \mid d (x', x) < \varepsilon , x \in R^m \},$$

is contained in X. This indicates that x′ is an interior point of X. By this contradiction, f(x′) must be a singleton.

Now we can show that x′ is an interior point of P_i by the induced topology of X. Let us suppose that x′ is a boundary point of P_i. Then there exists a Cauchy sequence $\{ x_n \}_{n \in N}$ in X such that $x_n \notin P_i$ for all $n \in N$ and that its limit point is x′. We choose arbitrarily y_n in $f(x_n)$ and define z_n as follows:

1. z_n is a convex combination of (x′, y′) and (x_n, y_n).
2. The distance between z_n and (x′, y′) is ε/i.

Such $z_n \in R^{m+n}$ always exists for every $n \in N$ because $x_n \notin P^i$, $(x_n, y_n) \in G(f)$, $(x_n, y_n) \notin A_i$ and $(x_n, y_n) \notin B_i$. Then $\{ z_n \}_{n \in N}$ is a bounded sequence in R^{m+n} and has a Cauchy subsequence. We denote its limit point by (x′, y″). Since $z_n \in G(f)$ for all $n \in N$, (x′, y″) belongs to the closure of G(f). By the strict convexity of G(f), for any λ' with $0 < \lambda' < 1$,

$$\lambda' (x', y') + (1 - \lambda') (x', y'')$$

is an interior point of G(f) and $\lambda' y' + (1 - \lambda')y''$ belongs to f(x′). We know that $y'' \neq y'$ since the distance between y″ and y′ is $\varepsilon/i > 0$. Therefore $\lambda' y' + (1 - \lambda') y'' \neq y'$ for any λ' with $0 < \lambda' < 1$. This contradicts the fact that f(x′) is a singleton.

We know now that x′ is an interior point of P_i . From this fact, we can prove 2) by similar way to our proof of 1).

QED

A correspondence with convex but not strictly convex graph is not always l.h.c.

Example Let X be a (closed) disk in R^2 whose boundary contains { (0, 0) } and X′ be $\{ (x, 0) \in R^3 \mid x \in X \}$. We define G as the convex hull of $\{ (0, 0, 1) \} \cup X'$. Then $f(x) \equiv \{ y \mid (x, y) \in G \}$ defines a correspondence f: X → R whose graph, G, is weakly convex. However this correspondence is not l.h.c. at $(0, 0) \in X$. In fact, we can take a Cauchy sequence, $\{ x_n \}_{n \in N}$, on the boundary of X which converges to (0, 0) and $f(x_n) = \{ 0 \}$ for all n except that f(0, 0) = [0, 1].

Dutta and Mitra 1989 gives another example.

We can generalize our proposition 2.1.1) as follows.

Proposition 2.2 We consider sets $X \subset R^m$, $Y \subset R^n$, and a correspondence f: X → Y. If for any point x′ in the interior of X and for any point y′ in f(x′), there exists a neighborhood V(x′) of x′, and a neighborhood U(y′) of y′ such that the correspondence h:V(x′) → U(y′) defined by $h(x) = f(x) \cap U(y')$ has convex graph, then f is l.h.c. at any point in the interior of X.

Proof Let x′ be an interior point of X and $\{ x_n \}_{n \in N}$ be a Cauchy sequence in X which converges to x′. For a point $y' \in f(x')$, $h(x) = f(x) \cap U(y')$ with convex graph is l.h.c. at x′ by proposition 2.1.1)). There exists a sufficiently large number M and Cauchy sequence $\{ y_n \}_{n \in N}$ in Y such that $y_n \in h(x_n) \subset f(x_n)$ for $n \geq M$ and $\{ y_n \}_{n \in N}$ converges to y′.

QED

When we specialize X = V(x′) and Y = U(y′), then proposition 2.2 turns out proposition 2.1.1).

The following corollary is a direct consequence of proposition 2.2.

Corollary 2.1 Consider $X \subset R^m$ and $Y \subset R^n$. Then a correspondence f:X → Y with open graph is l.h.c.

3. Upper Hemi-Continuity

Definition 3.1 Let X, Y be two topological spaces and f: X → Y be a correspondence. We say f is 'upper hemi-continuous (and abbreviate this to u.h.c.) at x′ ∈ X' if, for every neighborhood U of f(x′), there exists a neighborhood V of x′ such that f(x) ⊂ U for every x ∈ V. The correspondence is called 'u.h.c.' if it is u.h.c. at every x ∈ X.

Proposition 3.1 Let X be a subset of a linear metric space and Y be a subset of R^n. Then a compact-valued correspondence f: X → Y with strictly convex graph is u.h.c.

Proof Let X and Y be subsets of linear metric spaces L and M respectively, and f:X → Y be a correspondence. We consider the following three conditions:

① G(f) is strictly convex,

② for every x ∈ X, f(x) is closed by the topology of M and

③ for every x ∈ X, f(x) is bounded.

Then we will prove the following two results.

1) If f satisfies ① and ②, then G(f) is closed by the topology of the product space L × M.

2) If M = R^n, f satisfies ③, and G(f) is (weakly) convex and closed by the topology of the product space L × M, then f is u.h.c.

Proof of 1) Let { (x_n, y_n) } be a Cauchy sequence in G(f), and $(x_o, y_o) \in L \times M$ be the limit point of this sequence. All we have to do is to show that $(x_o, y_o) \in G(f)$. Since (x_o, y_o) is a point of the closure of G(f), by ①,

$$\lambda\ (x_o, y') + (1-\lambda)\ (x_o, y_o) \in G(f)$$

for any $y' \in f(x_o)$ and for any real number λ with $0 < \lambda < 1$. Then $\lambda y' + (1-\lambda)y_o \in f(x_o)$ and $\lambda y' + (1-\lambda)\ y_o \to y_o$ as $\lambda \to 0$. By ②, $y_o \in f(x_o)$, i.e. $(x_o, y_o) \in G(f)$.

Proof of 2) We suppose f is not u.h.c. at some point x′ ∈ X. Then there exists a neighborhood U of f(x′) such that $f(V) \cap U^c \neq \phi$ for any neighborhood V of x′ (Here, U^c is the complement of U. Without loss of generality, we can assume that U is an open subset of R^n. For whenever U′ is open by the topology of Y, there exists an open subset U of R^n such that $U' = U \cap Y$ and $Y - U' \subset R^n - U \equiv U^c$). Therefore we can take a sequence $\{ x_n \}_{n \in N}$ in X which converges to x′ and has the property :

$$f(x_n) \cap U^c \neq \phi \text{ for any } n \in N$$

(For example, we can take balls with center x′ and radius 1/n as neighborhoods of x′). Then, for any $n \in N$, we have an element y_n such that

$$y_n \in f(x_n) \cap U^c.$$

Since $f(x')$ is bounded by ③, there exists a bounded open set W with $U \supset W \supset f(x')$. Let us choose an arbitrary point y' in $f(x')$, and let W^b be the set of all boundary points of W. Then we have the fact :

$$y' \in f(x') \subset W \text{ and } y_n \in R^n - W \text{ for every } n \in N,$$

and there exists λ_n with $0 \leq \lambda_n \leq 1$ such that

$$\lambda_n y' + (1-\lambda_n) y_n \in W^b \text{ for every } n \in N$$

(For our purpose it is sufficient to take $\lambda_n = \inf \{ \lambda \mid \lambda y' + (1-\lambda) y_n \in W, 0 \leq \lambda \leq 1 \}$). Since W^b is bounded, the sequence $\{ \lambda_n y' + (1-\lambda_n) y_n \}_{n \in N}$ has a Cauchy subsequence $\{ \lambda_{n(i)} y' + (1-\lambda_{n(i)}) y_{n(i)} \}_{i \in N}$. Let y'' be the limit point of this sequence. Note that $y'' \in R^n - W$. According to n(i), we also get subsequences $\{ x_{n(i)} \}_{i \in N}$ of $\{ x_n \}_{n \in N}$, $\{ y_{n(i)} \}_{i \in N}$ of $\{ y_n \}_{n \in N}$ and $\{ \lambda_{n(i)} \}_{i \in N}$ of $\{ \lambda_n \}_{n \in N}$. Then we define $z_{n(i)}$ as follows :

$$z_{n(i)} \equiv \lambda_{n(i)} (x', y') + (1-\lambda_{n(i)}) (x_{n(i)}, y_{n(i)}).$$

It follows that $z_{n(i)} \in G(f)$ because $y' \in f(x')$, $y_{n(i)} \in f(x_{n(i)})$ and G(f) is convex (note that strict convexity of G(f) is unnecessary here). The limit point of the sequence $\{ z_{n(i)} \}_{i \in N}$ is (x', y''), because $\{ x_{n(i)} \}_{i \in N}$ converges to x' and $\{ \lambda_{n(i)} y' + (1-\lambda_{n(i)}) y_{n(i)} \}_{i \in N}$ converges to y''. Since

$$z_{n(i)} \in (X \times (R^n - W)) \cap G(f)$$

and G(f) is closed, (x', y'') belongs to $(X \times (R^n - W)) \cap G(f)$. Therefore

$$y'' \in R^n - W \text{ and } y'' \in f(x')$$

which contradicts our assumption $W \supset f(x')$.

QED

According to 1) and 2) in the proof of proposition 3.1, we have the following two corollaries.

Corollary 3.1 Let X and Y be subsets of linear metric spaces and $f: X \rightarrow Y$ be a correspondence. If the graph, G(f), of f is strictly convex and f(x) is closed for all $x \in X$, then G(f) is closed.

Figure 1

$X = Y = [\alpha, \ \beta] \subset R$
$f(x) = \{c\}$ if $x = \alpha$
$f(x) = [a, \ b]$ if $x \neq \alpha$

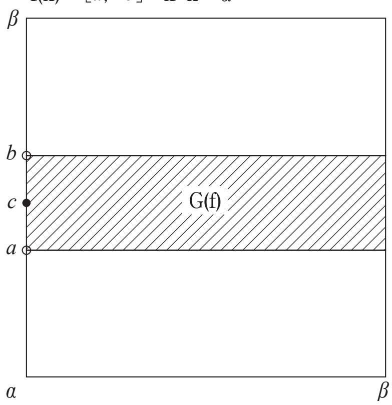
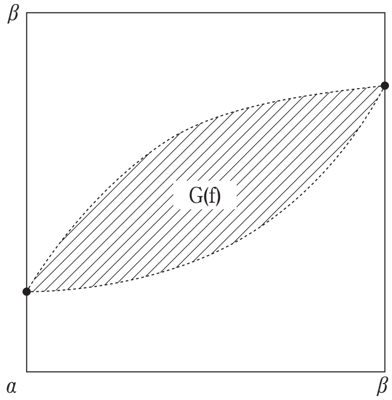


Figure 2

$X = Y = [\alpha, \ \beta] \subset R$
f(x) is open at x with $\alpha < x < \beta$



Corollary 3.2 Let X be a subset of a linear metric space, Y be a subset of R^n and $f{:}X \to Y$ be a correspondence. If f(x) is bounded for all $x \in X$ and the graph of f is (weakly) convex and closed, then f is u.h.c.

We can not remove conditions ① and ② . Figure 1 gives an example of f; f satisfies ② , G(f) is weakly convex instead of ① and f is not u.h.c. at α . Figure 2 gives an example of f; f satisfies ① , does not satisfy ② and is not u.h.c. at $x \in (\alpha, \beta)$.

Proposition 3.2 Let X and Y be subsets of R^m and R^n respectively, f: $X \to Y$ be a compact-valued correspondence, and G(f) be the graph of f. If G(f) is convex, then f is u.h.c. at any interior point of X.

We will prove this proposition by using the following two lemmas.

Lemma 3.1 Let G be a convex set in R^{m+n}, and we define

$$X \equiv \{ x \in R^m \mid \exists \ y \in R^n \ ((x, y) \in G) \} \qquad \text{(the projection of G on } R^m \text{).}$$

If G has an interior point, and x' is an interior point of X, then there exists a $y' \in R^n$ such that (x', y') is an interior point of G.

Lemma 3.2 Let G be a convex set in R^n, G^a be the closure of G and G^i be the interior of G (then the interior of G^a is denoted by G^{ai}). If G has an exterior point, then $G^{ai} = G^i$.

Proof of Lemma 3.1 Since x' is an interior point of X, there exists a neighborhood U of x' such that $U \subset X$. Let (x_1, y_1) be an interior point of G. If $x_1 = x'$, there is nothing to prove. So we assume that $x_1 \neq x'$. Let us pick up an element x_2 from the set

$$\{ \alpha x_1 + (1-\alpha) x' \mid \alpha \in R \} \cap U$$

with $\alpha < 0$. Then

$$x' = \lambda^* x_2 + (1-\lambda^*) x_1,$$

for some λ^* with $0 < \lambda^* < 1$.

Since $x_2 \in X$, we can find $y_2 \in R^n$ such that $(x_2, y_2) \in G$. Since (x_1, y_1) is an interior point of G, we can take a ball B in R^{m+n} with center (x_1, y_1) and sufficiently small radius with $B \subset G$.

Let us define the set :

$$A \equiv \{ \lambda (x_2, y_2) + (1 - \lambda) (x, y) \mid 0 \leq \lambda \leq 1, (x, y) \in B \}.$$

Since $(x_2, y_2) \in G$, $B \subset G$ and G is convex, A is contained in G. Now we consider

$$y' = \lambda^* y_2 + (1 - \lambda^*) y_1.$$

Then it is easily seen that the following point in R^{m+n}

$$(x', y') = \lambda^* (x_2, y_2) + (1-\lambda^*) (x_1, y_1)$$

belongs to A, and a ball with center (x', y') and sufficiently small radius is contained in A. This means that (x', y') is an interior point of G.

QED

Proof of Lemma 3.2 It suffices to show that $G^{ai} \subset G^i$. Suppose there exists $x \in R^n$ such that $x \in G^{ai}$ and $x \notin G^i$. Since x is a boundary point of G and G has an exterior point, by the separation theorem we can find a hyperplane H with the following property :

H separates R^n into two subsets S and S^c ($S^c = R^n - S$) where
$x \in H$, $H \subset S$ and $G \subset S$.

Since $x \in H$, any neighborhood of x has an element of S^c. On the other hand, since $x \in G^{ai}$, some neighborhood of x must be contained in G^a and does not have any exterior point of G. This is a contradiction.

QED

Proof of Proposition 3.2 We showed that the point (x', y'') in the proof of proposition 3.1 belongs to $G(f)^a$ ($G(f)^a$ is the closure of G(f)) by using (weak) convexity of G(f), without using strict convexity of G(f). But we used strict convexity of G(f) in order to show that G(f) is closed then $(x', y'') \in G(f)$ which has completed the proof of proposition 3.1 Therefore we will prove that $(x', y'') \in G(f)$ without using strict convexity but assuming that G(f) is (weakly) convex and x′ is an interior point of X. We also use the fact $(x', y'') \in G(f)^a$.

We can assume that G(f) has an interior point without loss of generality (If G(f) has no interior point, G(f) is a subset of a hyperplane in R^{m+n}. Without loss of generality, we can assume that the dimensions of X and Y are, respectively, m and n). When x′ is an interior point of X, by lemma 1 there exists $y' \in f(x')$ such that (x', y') is an interior point of G(f). Suppose that the point (x', y'') in the proof of proposition 3.1 is not an element of G(f). Let us define the set

$$C \equiv \{ \lambda y' + (1-\lambda)\, y'' \mid 0 \leq \lambda \leq 1 \} \cap f(x').$$

Since $y'' \notin f(x')$ and f(x′)is compact, there exists $y_o \in C$ which is the nearest point from y″ and we put

$$y_o = \lambda^* y' + (1 - \lambda^*)\, y''.$$

Since $y'' \notin f(x')$, λ^* is positive and we can take λ^{**} with $0 < \lambda^{**} < \lambda^*$.

Since (x', y') is an interior point of G(f), a ball $B \subset R^{m+n}$ with center (x', y') and sufficiently small radius is contained in G(f). Let us define the set A as

$$A \equiv \{ \lambda (x, y) + (1-\lambda)\, (x', y'') \mid 0 \leq \lambda \leq 1, (x, y) \in B \}.$$

It is well known that when a set is convex, the closure of the set is also convex. Therefore $G(f)^a$ is convex. Since $(x', y'') \in G(f)^a$ and $B \subset G(f) \subset G(f)^a$, A is contained in $G(f)^a$.

It is easily seen that the point

$$\lambda^{**} (x', y') + (1-\lambda^{**})\, (x', y'')$$

is an interior point of A. Therefore this is an interior point of $G(f)^a$ because $A \subset G(f)^a$. By lemma 2, this is also an interior point of G(f). This means that

$$\lambda^{**} y' + (1 - \lambda^{**})\, y'' \in f(x') \text{ and } 0 < \lambda^{**} < \lambda^*.$$

Then this contradicts our choice of λ^*. Therefore $(x', y'') \in G(f)$.

QED

Corollary 3.3 We consider sets $X \subset R^m$, $Y \subset R^n$ and a compact-valued correspondence f: X → Y with convex graph G(f). If x′ is an interior point of X and (x′, y″) is a boundary point of G(f), then (x′, y″) belongs to G(f).

4. Continuity

Definition 4.1 A correspondence f: X → Y is continuous if it is both l.h.c. and u.h.c.

Combining proposition 2.1.2) and proposition 3.1, we get the following proposition.

Proposition 4.1 We consider sets $X \subset R^m$ and $Y \subset R^n$. Then a compact-valued correspondence f: X → Y with strictly convex graph is continuous.

Combining proposition 2.1.1) and proposition 3.2, we have the following proposition.

Proposition 4.2 We consider sets $X \subset R^m$ and $Y \subset R^n$. Then a compact-valued correspondence f: X → Y with convex graph is continuous on the interior of X.

5. On some applications

Stokey and Lucas showed a misleading proposition in their first version (1989 p.61 theorem 3.5). After the pointing out by the author (1991), it was corrected as follows:

Let f: X → Y be a correspondence, and let G(f) be the graph of f. Suppose that G(f) is convex ; that for any bounded set $X' \subset X$, there is a bounded set $Y' \subset Y$ such that $f(x) \cap Y' \neq \phi$, all $x \in X'$. Then f is l.h.c. at every interior point of X.

In their first version, it used to have no the word ‘interior’ and the proof was incomplete. We have a counter example in section 2. Their revised theorem above is useful. However, our proposition 2.1.1) is more general than their theorem. Actually, the condition ‘that for any bounded set $X' \subset X$, there is a bounded set $Y' \subset Y$ such that $f(x) \cap Y' \neq \phi$, all $x \in X'$’ is unnecessary.

Appling our theorem 4.1, we get a modification of the maximum theorem of Berge 1963.

Proposition 5 We consider sets $X \subset R^m$ and $Y \subset R^n$. If f: X → R is a continuous function and g: X → Y is a compact-valued correspondence with strictly convex graph, then the maximum value,

M, defined by $M(x) = \max\{ f(y) \mid y \in g(x) \}$, is continuous on X and the correspondence of maximizers, h: $X \to Y$, defined by $h(x) = \{ y \mid y \in g(x), f(y) = M(x) \}$ is u.h.c.
Walker 1979 presents another modification.

It seems that our results are applicable to economic analyses such as general equilibrium analysis, dynamic programming, linear programming and especially the theory of intertemporal allocation.

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